5

Frequency Domain Analysis of Signals (Discrete Fourier Transform)

Discrete time signals can be analyzed or decomposed into a series of sines and cosines. This representation is called the discrete Fourier transform (DFT) of the signal; it is reversible and no information is lost. The DFT underlies most signal processing strategies, facilitates the interpretation of signals, enhances the characterization of systems, and improves the efficiency of algorithms. However, there are several inherent assumptions and limitations in this transformation.

Why are sines and cosines selected to analyze signals and systems? There are two reasons. First, sines and cosines are ***orthogonal functions*** and form a base for the analysis of signals, as discussed in this chapter. Second, sines and cosines are ***eigenfunctions*** for **LTI** systems; this will be the starting point for Chapter 6.

5.1 ORTHOGONAL FUNCTIONS – FOURIER SERIES

Two functions are orthogonal in the interval [a, b] if

$$\int_a^b f_u(t) \cdot \bar{f}_v(t)\, dt = \begin{cases} 0 & \text{if } u \neq v \\ c & \text{if } u = v \end{cases} \tag{5.1}$$

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where f_u and f_v are functions with real and imaginary components, $\bar{f}$ indicates complex conjugate of the function f, and c is any number different than zero.

Given a sinusoid of circular frequency $\omega = 2\pi/T$, its u-th harmonic is another sinusoid with circular frequency $\omega = u \cdot (2\pi/T)$, where u is an integer. Harmonics fulfill the orthogonality property; therefore, the following relations hold:

$$\int_0^T \sin\left(\frac{2\pi}{T}t\right) \cdot \sin\left(u\frac{2\pi}{T}t\right) \cdot dt = \begin{cases} 0 & \text{if } u \neq 1 \\ \frac{T}{2} & \text{if } u = 1 \end{cases} \tag{5.2}$$

$$\int_0^T \cos\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right) \cdot dt = \begin{cases} 0 & \text{if } u \neq 1 \\ \frac{T}{2} & \text{if } u = 1 \end{cases} \tag{5.3}$$

$$\int_0^T \sin\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right) \cdot dt = 0 \qquad \text{for all } u \tag{5.4}$$

Invoking Euler's identities (Chapter 2), these equations show that complex exponential-are orthogonal as well (see solved problem at the end of this Chapter):

$$\int_0^T e^{j\cdot\left(\frac{2\pi}{T}t\right)} \cdot e^{-j\cdot\left(u\frac{2\pi}{T}t\right)} \cdot dt = \begin{cases} 0 & \text{if } u \neq 1 \\ T & \text{if } u = 1 \end{cases} \tag{5.5}$$

The integral equation used to determine the orthogonality of two functions is equivalent to the equation used to determine the value of cross-correlation for zero time shift ($\tau = 0$ in continuous time). Hence, orthogonality concepts support the utilization of cross-correlation to identify frequency similarity between two signals (Chapter 4).

5.1.1 Fourier Series

The orthogonality of harmonics suggests that these functions form a ***base*** in the open interval [0, T[. Then, a continuous periodic function f(t) with period T can be expressed as a linear combination of sinusoids with frequencies that are multiples of the fundamental circular frequency $2\pi/T$. The summation is known as Fourier series. The value at discrete time t_i is

$$f_i = \sum_{u=-\infty}^{\infty} \left[a_u \cdot \cos\left(u\frac{2\pi}{T}t_i\right) + b_u \cdot \sin\left(u\frac{2\pi}{T}t_i\right)\right] \tag{5.6}$$

where the coefficients a_u and b_u are real.

5.1.2 An Intuitive Preview of the Fourier Transform

Imagine N points in the x–t Cartesian coordinates $\underline{x} = [x_0, x_1, x_2, x_3, \ldots]$. If a polynomial is least squares fitted through these points $p(t) = a + bx + cx^2 + dx^3 + \ldots$, we could call the set of coefficients $\underline{p} = [a, b, c, d, \ldots]$ the "polynomial transform of $\underline{x}$".

Likewise, one can least squares fit the Fourier series (Equation 5.6) to the signal $\underline{x}$. The set of coefficients $\underline{X} = [a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3 \ldots]$ is called the Fourier transform of $\underline{x}$, which is herein denoted with a capital letter. The subset made of all a-coefficients is called the "real" part $Re(\underline{X})$, whereas the subset of b-coefficients is called the "imaginary" part $Im(\underline{X})$. Each subset plotted versus the frequency counter u provides important information about the signal $\underline{x}$:

- The u-th value in $Re(\underline{X})$ is the amplitude of the cosine with frequency $u(2\pi/T)$ that is needed to form the signal $\underline{x}$.
- The u-th value in $Im(\underline{X})$ is the amplitude of the sine with frequency $u(2\pi/T)$ that is needed to form the signal $\underline{x}$.

where the fundamental period T as the length of the time window, so that the fundamental circular frequency is $2\pi/T$. Figure 5.1 shows a collection of simple signals and the corresponding real and imaginary parts of their Fourier transform obtained by fitting the Fourier series to the signals. The signals are simple and Fourier transforms are identified by visual inspection and comparison with Equation 5.6. A few important observations follow:

- A constant signal, $x_i =$ constant for all i, has no oscillations; therefore, all terms for $u > 0$ are null: $a_{u>0} = 0$ and $b_{u>0} = 0$. For $u = 0$, $\cos(0) = 1$, and the first real coefficient a_0 takes the value of the signal. On the other hand, $\sin(0) = 0$, and any value for the first imaginary coefficient b_0 could be used; however, $b_0 = 0$ is typically assumed. For example, fitting the Fourier series to $\underline{x} = [7, 7, 7, 7, 7, \ldots]$ results in $Re(X) = [7, 0, 0, 0, 0, \ldots]$ and $Im(X) = [0, 0, 0, 0, 0, \ldots]$, as shown in Figure 5.1a.
- The Fourier transform of a single frequency cosine signal is an impulse in $Re(X)$, whereas the transform of a single frequency sine is an impulse in $Im(X)$. For example, if a sine signal with amplitude 7 fits three times in the time window, then the Fourier transform obtained by fitting Equation 5.6 is an impulse corresponding to the third harmonic in the imaginary component, $b_3 = 7$, and $Re(X) = [0, 0, 0, 0, 0, 0, \ldots]$ and $Im(X) = [0, 0, 0, 7, 0, 0, \ldots]$ as shown in Figure 5.16.
- Because the Fourier series is a summation, superposition is implied, and the cases in Figure 5.1d and e are readily computed.

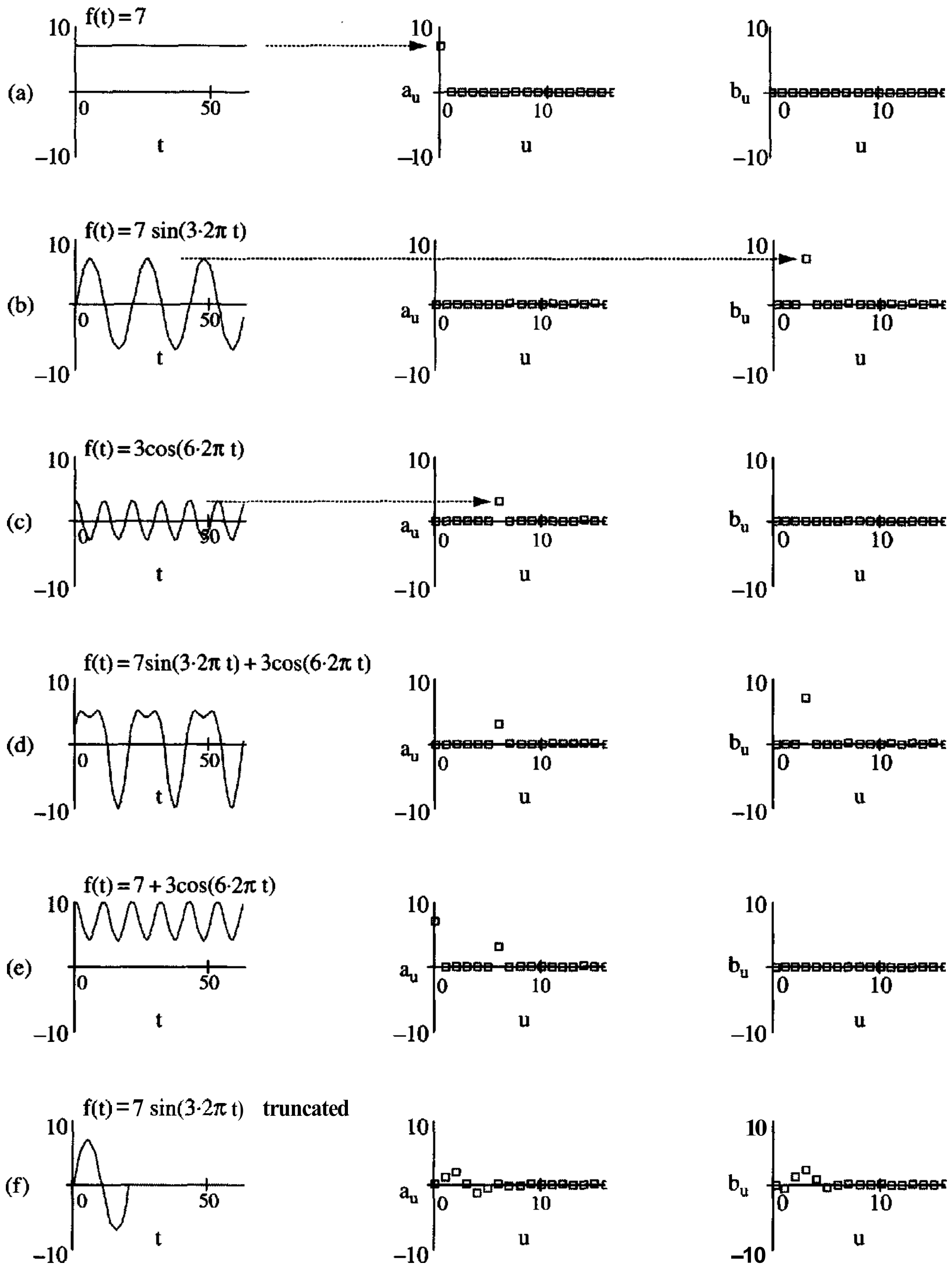


Figure 5.1 Simple signals and the corresponding real (cosine) and imaginary (sine) components of the fitted Fourier series. Note that the truncated sinusoid requires additional frequency components to synthesize the signal

- What is the Fourier transform of a signal duration T with a one-cycle sinusoid duration T/3, shown in Figure 5.1f? A good initial guess is to assume that b_3 will not be zero. Furthermore, there must be other nonzero real and imaginary components; otherwise, the sinusoid would be present throughout the duration of the signal.

This intuitive preview suggests a robust interpretation of the Fourier transform: it is curve fitting the signal a series of cosines (real part) and sines (imaginary part). However, there are several subtleties. For example, note that the signal $\underline{x}$ exists from time zero to T, that is $0 \le t_i < T$. However, the sinusoids that are used to fit the signal $\underline{x}$ exist from "the beginning of time till the end of time, all the time", that is $-\infty < t < +\infty$. The implications of discretization are explored in the next sections.

5.2 DISCRETE FOURIER ANALYSIS AND SYNTHESIS

There are four types of Fourier time-frequency transforms according to the continuous or discrete representation of the information in each domain: continuous-continuous, continuous-discrete, discrete-continuous, and discrete-discrete. Current engineering and science applications invariably involve discrete time and frequency representations. Consequently, only the case of discrete-discrete transformation is considered.

There is an immediate and most relevant consequence of selecting discrete time and frequency representations: The discrete time and frequency Fourier *transform* presumes periodic signals. In other words, any aperiodic signal $\underline{x}$ with N points $[x_0, \ldots, x_{N-1}]$ is automatically assumed periodic with fundamental period $T = N \cdot \Delta t$. A schematic representation is shown in Figure 5.2.

5.2.1 Synthesis: The Fourier Series Rewritten

The Fourier series in Equation 5.6 is rewritten to accommodate the discrete nature of the signals in time and frequency domains, and the inherent periodicity associated with the discrete representation. The sequence of changes is documented next.

Change #1: Exponentials

Sines and cosines are replaced for complex exponentials by means of Euler's identities with complex coefficients X_u (Chapter 2):

$$f(t) = \sum^{+\infty} X_u \cdot e^{j \cdot \left(u \frac{2\pi}{T} t\right)} \tag{5.7}$$

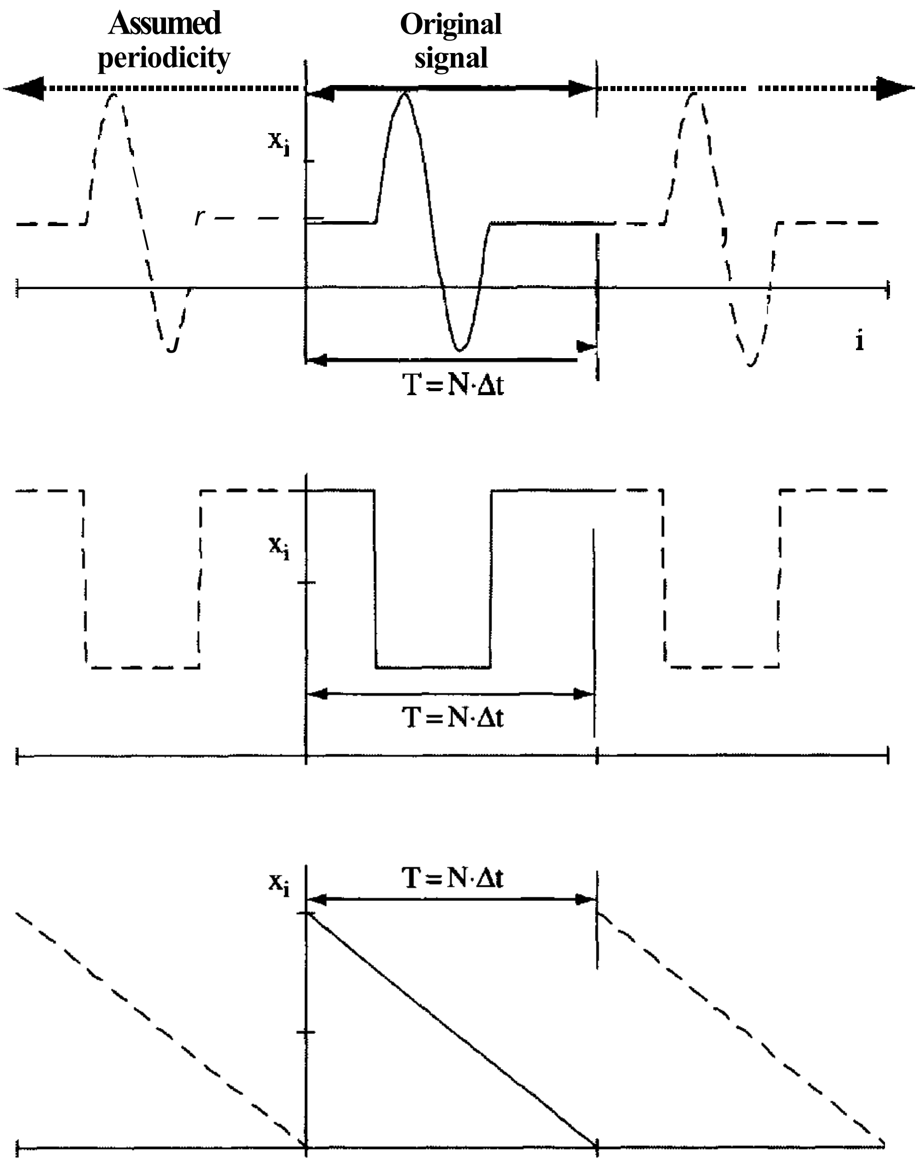


Figure 5.2 The discrete time and frequency Fourier transform assumes periodicity. Therefore, aperiodic signals are converted to periodic signals. The continuous line represents the captured signal. The dotted lines are the presumed periodic repetition from time $-\infty$ to $+\infty$

Change #2: Discrete Time

The inherent periodicity of a discrete time signal $\underline{x}$ is $T = N \cdot \Delta t$ and discrete time time is $t_i = i\Delta t$. Then, Equation 5.7 becomes

$$x_i = \sum^{+\infty} X_u \cdot e^{j\cdot\left(u \frac{2\pi}{N\cdot\Delta t} i\cdot\Delta t\right)} = \sum^{+\infty} X_u \cdot e^{j\cdot\left(u \frac{2\pi}{N} i\right)} \tag{5.8}$$

Change #3: Nyquist Criterion

The highest frequency that can be resolved from a discrete time signal is the Nyquist frequency $1/(2 \cdot \Delta t)$, as shown in Chapter 3. Therefore, the highest frequency of any harmonic to be included in the series is $u_{max} \cdot (1/T) = 1/(2 \cdot At)$. Replacing $T = N \cdot At$, the discrete time Fourier series need not extend beyond $u_{max} = N/2$. Keeping N summation terms, from $-N/2$ to $(N/2) - 1$,

$$x_i = \sum_{u=-\frac{N}{2}}^{\frac{N}{2}-1} X_u \cdot e^{j \cdot \left(u \frac{2\pi}{N} i\right)} \tag{5.9}$$

Change #4: Shift in Summation Limits

The complex exponential does not change if either u or $u + N$ appear in the exponent because $e^{j2\pi} = e^{j2\pi^N} = 1$. Then the summation limits are shifted while keeping N-terms in the summation. In particular, Equation 5.9 can be written as

$$x_i = \sum_{u=0}^{N-1} X_u \cdot e^{j \cdot \left(u \frac{2\pi}{N} i\right)} \tag{5.10}$$

where negative frequencies are avoided. The fact that the summation limit goes above $N/2$ does not imply that higher frequencies are extracted from the discrete signal. This is just a mathematical effect that will be discussed further in the text. An important conclusion from this analysis is that the ***Fourier series for discrete time periodic signals is finite.***

5.2.2 Analysis: Computing the Fourier Coefficients

Fourier coefficients X_u can be obtained by least squares fitting the signal $\underline{x}$ with the Fourier series in Equation 5.10: given the array $\underline{x}$, identify each coefficient X_u so that the total square error E between measured values x_i and predicted values $x_i^{<pred>}$ is minimized, $\min[E = \Sigma(x_i - x_i^{<pred>})^2]$. When the fitting is complete, the residual is $E = 0$. (There may be some numerical noise. See solved problems in Chapter 3.)

A better alternative is to call upon the orthogonality property of harmonics to identify how much the signal $\underline{x}$ (N points sampled with an interval At) resembles a given sinusoid of frequency $\omega_u = u \,.\, 2\pi/(N \,.\, At)$. Following this line of thought, the Fourier coefficients are computed as the zero-shift value of the cross-correlation:[1]

$$X_u = \sum_{i=0}^{N-1} x_i \cdot e^{-j \cdot \left(u \frac{2\pi}{N} i\right)} \quad (5.11)$$

Note that Equation 5.11 is a summation in the time index i, whereas Equation 5.10 is a summation in the frequency index u. The Fourier coefficient $Re(X_0) = \sum x_i$ captures the ***static component*** of the signal (zero-offset or DC-offset) and the zero frequency imaginary coefficient is assumed $Im(X_0) = 0$. The array $\underline{X}$ formed with the complex Fourier coefficients X_u is the "discrete Fourier transform" or frequency domain representation of the discrete time signal $\underline{x}$. The magnitude of the Fourier coefficient X_u relates to the magnitude of the sinusoid of frequency $\omega_u = u \cdot 2\pi/T$ that is contained in the signal with phase ϕ_u

$$|X_u| = \sqrt{[Re(X_u)]^2 + [Im(X_u)]^2} \quad \textit{amplitude} \quad (5.12)$$

$$\varphi_u = \tan^{-1}\left(\frac{Im(X_u)}{Re(X_u)}\right) \quad \textit{phase} \quad (5.13)$$

5.2.3 Selected Fourier Pair

The analysis equation and its corresponding synthesis equation form a "Fourier pair". From Equations 5.10 and 5.11,

$$X_u = \sum_{i=0}^{N-1} x_i \cdot e^{-j \cdot \left(u \frac{2\pi}{N} i\right)} \quad \textit{analysis equation: time} \rightarrow \textit{frequency} \quad (5.14)$$

$$x_i = \frac{1}{N} \sum_{u=0}^{N-1} X_u \cdot e^{j \cdot \left(u \frac{2\pi}{N} i\right)} \quad \textit{synthesis equation: frequency} \rightarrow \textit{time} \quad (5.15)$$

The normalization factor 1/N is added in the synthesis equation to maintain energy content in time → frequency → time transformations.

There are different Fourier pairs available in computer software and invoked in the literature. This Fourier pair is notably convenient in part owing to the

[1] The Fourier transform and the Laplace transform in continuous time are:

$$\text{Fourier: } X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j \cdot \omega \cdot t} dt \qquad \text{Laplace: } X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-s \cdot t} dt$$

where s is the complex variable $s = \sigma + j \cdot \omega$. When $\sigma = 0$, the Laplace transform becomes the Fourier transform. The z-transform is the discrete time equivalent of the Laplace transform.

parallelism between the analysis and the synthesis equations. (Other advantages will be identified in Chapter 6.) If the DFT is implemented with a given analysis equation, the inverse **DFT (IDFT)** must be computed with the corresponding synthesis equation in the pair. Table 5.1 summarizes the Fourier pair and related expressions.

The **DFT** of a one-dimensional **(1D)** signal in time involves the frequency $\omega = 2\pi/T$ and its **harmonics.** If the parameter being monitored varies along a spatial coordinate ℓ, the wave number $\kappa = 2\pi/\lambda$ is used instead. Analogous to signals in time, the maximum wavelength λ that is captured in the discrete record depends on the sampling interval $\Delta\ell$ and the number of points N so that $\lambda = N \cdot \Delta\ell$, and the exponent $u \cdot \omega \cdot t$ in the complex exponential becomes

$$u \cdot \frac{2\pi}{\lambda} \cdot \ell = u \cdot \frac{2\pi}{N \cdot \Delta\ell} \cdot i \cdot \Delta\ell = u \cdot \frac{2\pi}{N} \cdot i \qquad \textit{in space} \qquad (5.16)$$

Therefore, the formulation presented earlier is equally applicable to spatial variables.

Table 5.1 Summary: discrete Fourier transform pair and related expessions

Analysis *(from time → to frequency)*	**Synthesis** *(from frequency → to time)*
$X_u = \sum_{i=0}^{N-1} x_i \cdot e^{-j\left(u \frac{2\pi}{N} i\right)}$	$x_i = \frac{1}{N} \sum_{u=0}^{N-1} X_u \cdot e^{j \cdot \left(u \frac{2\pi}{N} i\right)}$
Static component:	$X_0 = \sum_i x_i$
Magnitude:	$\lvert X_u \rvert = \sqrt{[Re(X_u)]^2 + [Im(X_u)]^2}$
Phase:	$\varphi_u = \tan^{-1}\left[\frac{Im(X_u)}{Re(X_u)}\right]$
Parseval's identity:	$\sum_{i=0}^{N-1} x_i^2 = \frac{1}{N} \cdot \sum_{u=0}^{N-1} \lvert X_u \rvert^2$

The following expressions are worth highlighting:

$t_i = i \cdot At$	$T = N \cdot \Delta t$
$f_{min} = \frac{1}{T}$	$f_{max} = \frac{1}{2 \cdot \Delta t}$
$f_u = u\frac{1}{T} = u\frac{1}{N \cdot \Delta t}$	$\omega_u = 2\pi f_u = u\frac{2\pi}{T} = u\frac{2\pi}{N \cdot \Delta t}$

Note:
The physical dimensions are the same in both domains.
Summations in "u" can be reduced to (N/2)+1 terms by recalling the symmetry and periodicity properties. When the summation is extended from u = 0 to u = N − I the operation is called "double sided". When the summation is extended from u = 0 to N/2, the operation is called "single sided".

5.2.4 Computation - Example

In 1965, J. Tukey and J. Cooley published an algorithm for the efficient implementation of the DFT. This algorithm and other similar ones developed since are known as the "fast Fourier transform" (FFT). Maximum computational efficiency is attained when the signal length is a power of 2, $N = 2^r$, where r is an integer.

Signal analysis and synthesis are demonstrated in Figure 5.3. The aperiodic tooth signal in Figure 5.3a is transformed to the frequency domain. (Recall that the discrete time and frequency representation presumes this signal repeats itself.) Both real and imaginary components are shown in Figures 5.3b and c. Observe that the static component is equal to Σx_i. The synthesis of the signal is incrementally computed by adding increasingly more terms in the Fourier series. Figures 5.2d–k show the evolution of the synthesized signal. The last synthesized signal in Figure 5.2k is identical to the original signal $\underline{x}$.

5.3 CHARACTERISTICS OF THE DISCRETE FOURIER TRANSFORM

The most important properties of the DFT are reviewed in this section. Exercises at the end of this chapter suggest the numerical verification of these properties.

5.3.1 Linearity

The Fourier transform is a sum of binary products, thus, it is distributive. Therefore, given two discrete time signals & and $\underline{y}$, and their Fourier transforms $\underline{X}$ and $\underline{Y}$

$$\left(a \cdot \underline{x} + b \cdot \underline{y}\right) \xrightarrow{DFT} (a \cdot \underline{X} + b \cdot \underline{Y}) \tag{5.17}$$

5.3.2 Symmetry

The cosine is an even function $\cos(u\theta) = \cos(-u\theta)$, whereas sine is odd $\sin(u\theta) = -\sin(-u\theta)$. Therefore, it follows from Euler's identities (Chapter 2) that the Fourier coefficient for the frequency index u is equal to the complex conjugate of the Fourier coefficient for $-u$

$$X_u = \overline{X_{-u}} \tag{5.18}$$

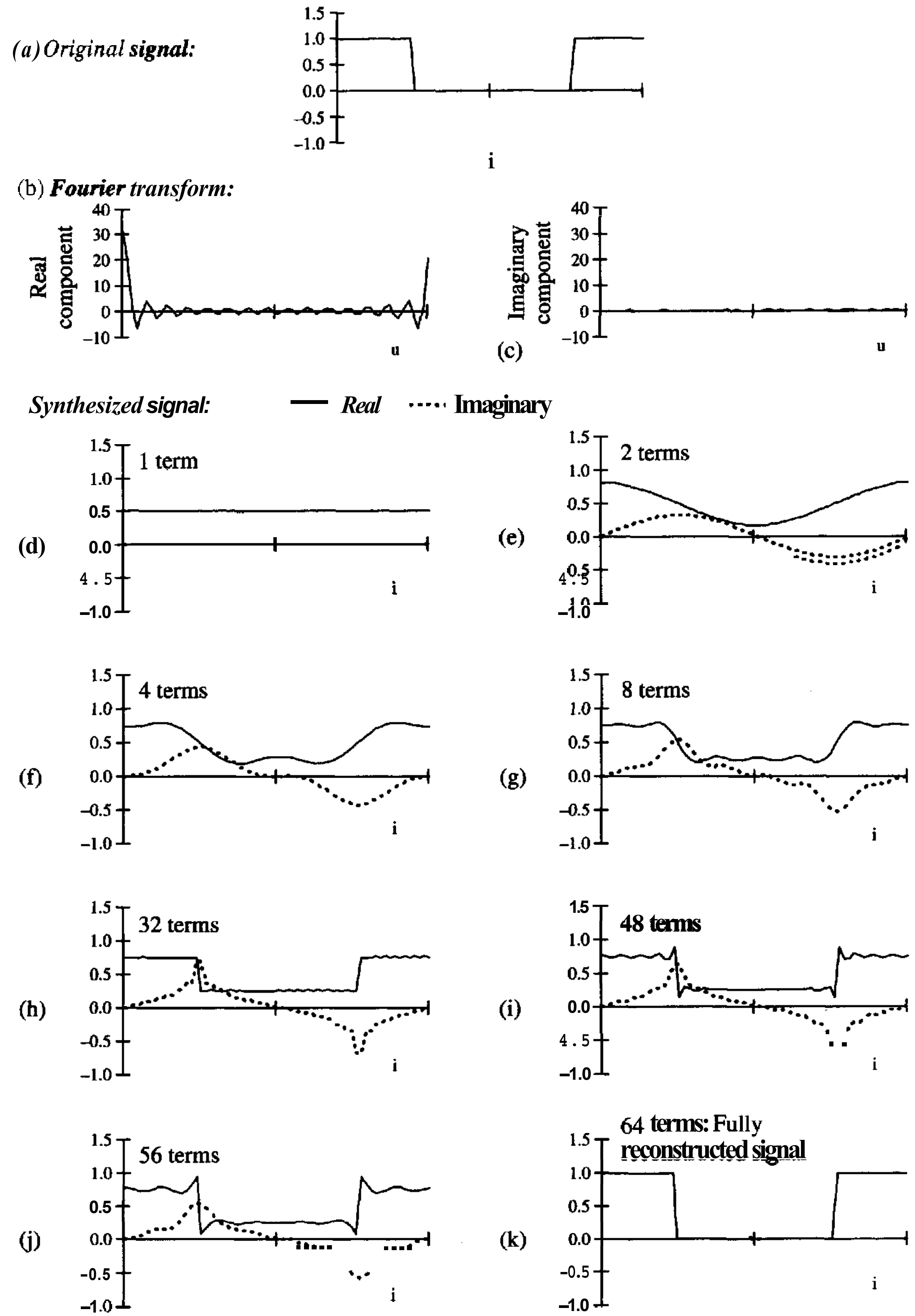


Figure 5.3 Analysis and synthesis: (a) original signal, N = 64; (b) and (c) analysis: real and imaginary components of the DFT; (d)–(k) synthesis: incremental reconstruction of the signal by adding an increasingly higher number of Fourier components

5.3.3 *Periodicity*

As invoked earlier in relation to Equation 5.10, the complex exponential for frequency $\omega_u = (u + N) \cdot (2\pi/N \cdot At)$ has the same values at discrete times t_i as an exponential with lower frequency $\omega_u = u(2\pi/N \cdot At)$. Therefore,

$$X_u = X_{u+N\cdot k} \tag{5.19}$$

where k is an integer. Therefore, ***the discrete time and frequency domain assumption inherently implies a periodic signal in time and in frequency,*** and the corresponding arrays in each domain repeat with periodicity:

$$T = N \cdot At \text{ (in time domain)} \qquad N\frac{2\pi}{T} = \frac{2\pi}{At} \text{ (in frequency domain)} \tag{5.20}$$

Figure 5.4 presents a discrete signal $\underline{x}$ and its discrete transform $\underline{X}$, and highlights the periodicities in time domain and frequency domains.

5.3.4 *Convergence - Number of Unknown Fourier Coefficients*

It would appear that there are N complex coefficients X_u; hence, $2 \cdot N$ unknowns. However, the periodicity and symmetry properties of the Fourier transform guarantee that $X_u = \overline{X_{N-u}}$, where the bar indicates complex conjugate. Furthermore, X_0 and $X_{N/2}$ are real. Then, the number of unknowns is reduced to N. Indeed, this must be the case: each value x_i permits writing one equation like Equation 5.15, and given that complex exponentials form a base, the number of unknown Fourier coefficients must be equal to the number of equations N. The following numerical example verifies these observations. Consider the time series $\underline{x} = [1, 0, 1, 1, 0, 1, 1, 2]$ with $N = 8$ elements. The DFT of $\underline{x}$ is obtained using Equation 5.14:

u	0	1	2	3	4	5	6	7
X_u	7	$1+j\cdot\sqrt{2}$	$-1+j\cdot\sqrt{2}$	$1+j\cdot\sqrt{2}$	-1	$1-j\cdot\sqrt{2}$	$-1-j\cdot\sqrt{2}$	$1-j\cdot\sqrt{2}$

Note that the array $\underline{X}$ fulfills the relation $X_u = \overline{X_{N-u}}$, and that $X_0 = 7$ and $X_{N/2} = -1$ are real; therefore, there are only N unknowns.

The fact that N values in the time domain are fitted with N Fourier coefficients in the frequency domain implies that there will be no convergence difficulties in the DFT of discrete time signals. (Convergence problems develop in continuous

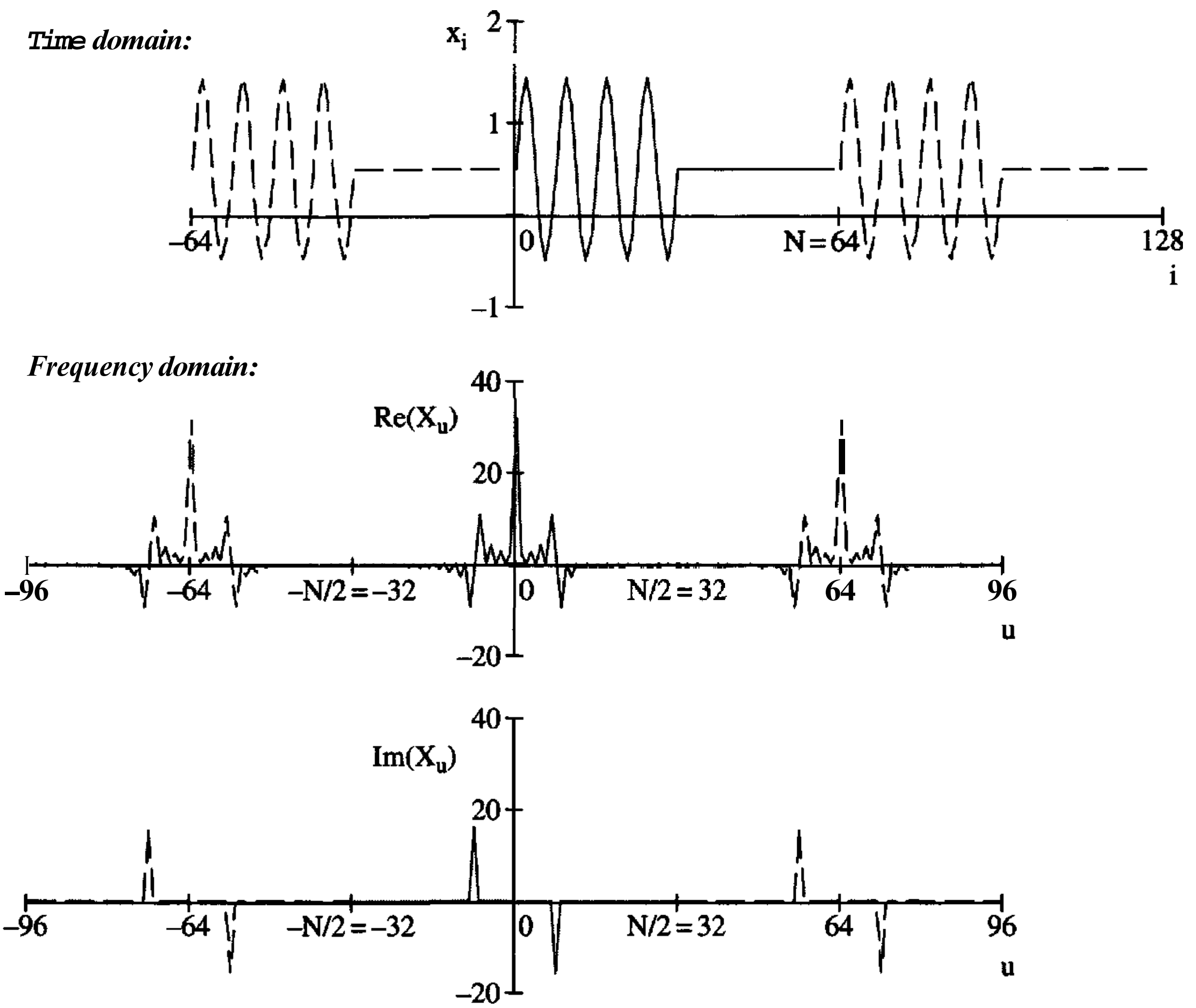


Figure 5.4 The DFT presumes the signal is periodic both in the time and the frequency domains. Observe the symmetry properties of real and imaginary components. The time series $\underline{x}$ has a DC offset, thus $\text{Re}(X_0) \neq 0$.

time signals around discontinuities. This is Gibb's phenomenon, and it manifests as ripples and overshoots near discontinuities.) In addition, the N information units available in the time domain are preserved in the frequency domain, as confirmed by the fact that $\underline{x} = \text{IDFT}[\text{DFT}(\underline{x})]$, indicating that there is no loss of information going from time to frequency and vice versa.

5.3.5 One-sided and Two-sided Definitions

The DFT was defined for the frequency index u that ranges from $u = -N/2$ to $(N/2) - 1$ or from $u = 0$ to $u = N - 1$. These are called ***two-sided definitions.*** Yet, there is no need to duplicate computations when $X_u = \overline{X_{N-u}}$: one does not physically measure negative frequencies, and cannot resolve above the Nyquist frequency. Therefore, ***one-sided definitions*** are established between $u = 0$ and

$u = N/2$. Two-sided definitions are advantageous in analytical derivations. However, one-sided definitions are computationally efficient. (See exercise at the end of this chapter.) To avoid confusion, derivations, computations, and examples in this text are obtained with two-sided definitions.

5.3.6 Energy

The energy in a signal $\underline{x}$ is the sum of the square of the amplitude of the signal at each point. Each Fourier coefficient X_u indicates the amplitude of the sinusoid of frequency $\omega_u = u \cdot 2\pi/T$ that is contained in the signal. Therefore, the energy in the signal is also computed from the Fourier coefficients, as prescribed in Parseval's identity,

$$\sum_{i=0}^{N-1} x_i^2 = \frac{1}{N} \cdot \sum_{u=0}^{N-1} |X_u|^2 \tag{5.21}$$

The plot of $|X_u|^2$ versus frequency is the ***autospectral density*** of the signal, also known as power spectral density. (Spectral values in one-sided computations are twice those corresponding to the two-sided definition except for the zero-frequency term.)

5.3.7 Time Shift

Consider a wave train propagating along a rod. The signal is detected with two transducers. If the medium is not dispersive or lossy, and the coupling between the transducers and the rod are identical, then the only difference between the signal $\underline{x}$ detected at the first transducer and the signal $\underline{y}$ detected at the second transducer is the wave travel time between the two points $r \cdot \Delta t$. For a single frequency ω sinusoid,

$$\text{if} \quad x_i = e^{j\omega i \Delta t}$$
$$\text{and} \quad y_i = x_{i-r} = e^{j\omega(i-r)\Delta t} - x_i e^{-j\omega r \Delta t} \tag{5.22}$$
$$\text{then} \quad Y_u = e^{-j\left(u\frac{2\pi}{N}r\right)} \cdot X_u$$

For the given travel time, the higher the frequency signal, the higher the phase shift. When phase is measured, computed arctan values can only range between $[\pi/2, -\pi/2]$, and proper "phase unwrapping" is required (Chapter 6).

5.3.8 Differentiation

The derivative of a continuous time sinusoid $x(t) = A \cdot \sin(\omega \cdot t)$ is $y(t) = \omega . A \cdot \cos(\omega \cdot t)$. In words, the derivative of a sinusoid implies a linear scaling of the amplitude by the frequency and a $\pi/2$ phase shift. The first derivative in discrete time $\underline{y}$ can be approximated by finite differences. The corresponding DFT is obtained by invoking the time shift property (Equation 5.22):

$$y_i = \frac{x_i - x_{i-1}}{\Delta t} \quad \text{then} \quad Y_u = \frac{1 - e^{-j\left(u\frac{2\pi}{N}\right)}}{\Delta t} X_u \tag{5.23}$$

The magnitude of the coefficient that multiplies X_u increases with u. Thus, this result predicts the magnification of high-frequency components when a differentiation transformation is imposed. This is in agreement with observations in the time domain whereby the derivative of a signal is very sensitive to the presence of high-frequency noise.

5.3.9 Duality

The parallelism between the analysis and synthesis equations in a Fourier pair (Equations 5.14 and 5.15, Table 5.1) leads to the property of duality. Before proceeding, notice that the exponents have the opposite sign in the Fourier pair; this means opposite phase: one is turning clockwise and the other counterclockwise, or in terms of time series, one is the tail-reverse version of the other. (For clarity, replace the exponentials for their trigonometric identities: a tail-reverse cosine is the same cosine; however, a tail-reversed sine becomes [−]sine, thus opposite phase.)

Now, consider the signal $\underline{x}$ shown in Figure 5.5a. The DFT of signal $\underline{x}$ computed with Equation 5.14 is shown in Figures 5.5b and c. Then, the analysis Equation 5.14 is used again to compute a second DFT but this time of $\underline{X}$, that is DFT[DFT($\underline{x}$)]. Figure 5.5c shows that the result is the original signal but in reversed order and scaled by N. In mathematical terms,

$$(x_0, x_{N-1} \ldots x_1) = \frac{1}{N} . \text{DFT}[\text{DFT}(x_0, x_1 \ldots x_{N-1})] \tag{5.24}$$

Duality is a useful concept in the interpretation of time and frequency domain operations and properties.

5.3.10 Time and Frequency Resolution

The time resolution is defined as the time interval between two consecutive discrete times; this is the sampling interval $\Delta t = t_{i+1} - t_i$. Likewise, frequency

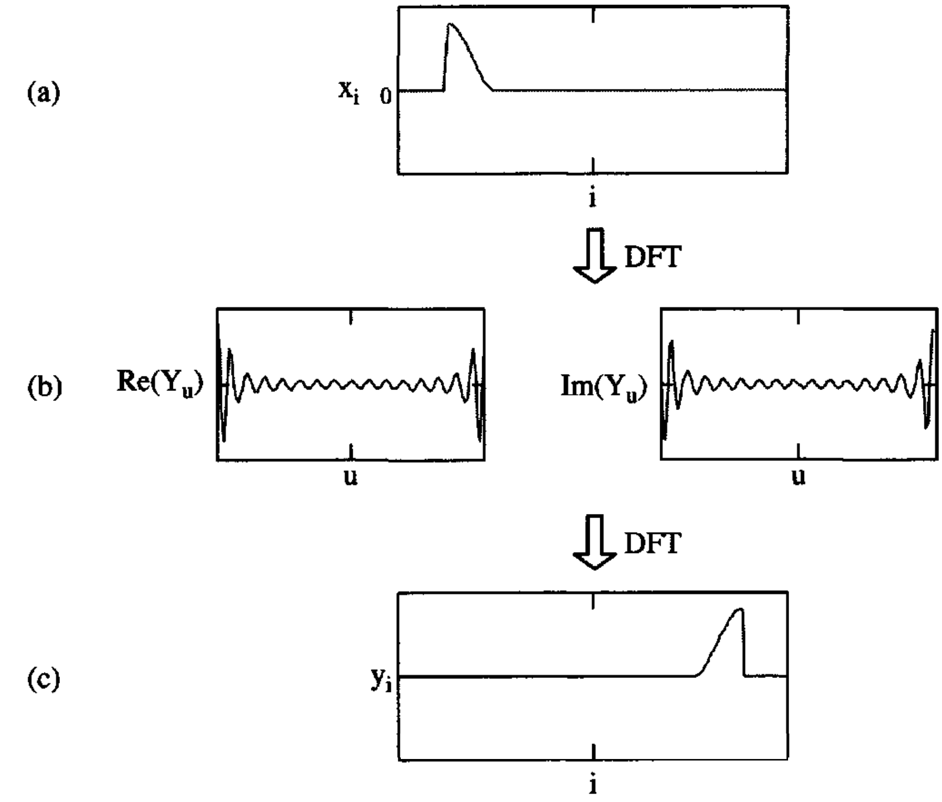


Figure 5.5 The duality property of the DFT: (a) the original signal x; (b) its discrete Fourier transformed to the frequency domain X; (c) the *forward* (not inverse) discrete Fourier transformation of X sends the series back to the time domain, but the signal appears tail-reversed

resolution is the frequency interval between two consecutive discrete frequencies $Af = f_{u+1} - f_u$, where each frequency f_u is the u-th harmonic of the first frequency $f_u = u \cdot f_1 = u/(N \cdot At)$. Then $Af = f_{u+1} - f_u = (u + 1 - u)/(N \cdot At)$:

$$\Delta f = \frac{1}{N \cdot \Delta t} \quad \text{that is} \quad N = \frac{1}{Af \cdot At} \tag{5.25}$$

This is known as the "uncertainty principle" in signal processing: ***when limited to N pieces of information, the resolution in frequency can only be improved at the expense of the resolution in time*** (see solved example at the end of this Chapter).

5.3.11 Time and Frequency Scaling

The length N of the array x can be reduced by decimation (removal of intermediate points) or increased by interpolation. Similar effects are obtained by varying the sampling interval At during A/D conversion: down-sampling or up-sampling. In either case, the total time duration of the signal remains the same. Consider a stationary continuous signal $x(t)$ sampled with two different sampling rates At and $\alpha \cdot \Delta t$:

Discrete time	Signal	Discrete frequency	DFT
$t_i = i \cdot \Delta t$	y_i	$\omega_u = u\frac{2\pi}{N \cdot \Delta t}$	Y_u
$t_k = k \cdot (\alpha \cdot \Delta t)$	z_k	$\omega_v = v\frac{2\pi}{M \cdot \alpha \cdot \Delta t}$	Z_v

The values z_i and z_k are equal at the same discrete time $t_i = t_k$; therefore, $i = k \cdot a$. Likewise, the values of Y_u and $\alpha \cdot Z_v$ are equal at the same discrete frequency $\omega_u = \omega_v$; therefore, $u = v/\alpha$. Thus,

$$\text{if} \quad y_{k \cdot \alpha} \quad \text{then} \quad \frac{1}{a} \cdot Y_{\left(\frac{v}{\alpha}\right)} \tag{5.26}$$

This result shows the inherent inverse relation between time and frequency. The factor $1/\alpha$ in the frequency domain reflects the selected Fourier pair. Down-sampling is restricted by the Nyquist frequency.

5.4 COMPUTATION IN MATRIX FORM

The summation of binary products in analysis and synthesis equations is equivalent to matrix multiplication, and the transformation $\underline{X} = \text{DFT}(\underline{x})$ implied in Equation 5.14 can be computed as:

$$\underset{[N,1]}{\underline{X}} = \underset{[N,N]}{\underline{\underline{F}}} \cdot \underset{[N,1]}{\underline{x}} \qquad \textit{Time} \rightarrow \textit{Frequency} \tag{5.27}$$

where each row in the Fourier transform matrix $\underline{\underline{F}}$ is the array of values that represents a complex exponential. In other words, the i-th element in the u-th row of $\underline{\underline{F}}$ is

$$F_{u,i} = e^{-j \cdot \left(u \frac{2\pi}{N} i\right)} \tag{5.28}$$

Note that u and i play the same roles in the exponent; therefore, the element $F_{u,i}$ is equal to the element $F_{i,u}$ and the matrix is symmetric $\underline{\underline{F}}^T = \underline{\underline{F}}$.

Similarly, the implicit operations in matrix multiplication apply to the synthesis equation or inverse Fourier transform. The elements in the inverse Fourier matrix $\underline{\underline{InvF}}$ have positive exponent, and the following equality holds (see Equation 5.15):

$$InvF_{u,i} = \frac{1}{N} e^{j\cdot\left(u\frac{2\pi}{N}i\right)} = \frac{1}{N}\overline{e^{-j\cdot\left(u\frac{2\pi}{N}i\right)}} = \frac{1}{N}\overline{F}_{u,i} \tag{5.29}$$

where the bar indicates complex conjugate. (Note: this is in agreement with the duality property, where the conjugate implies reversal.) Therefore, the inverse Fourier transform is

$$\underline{x} = \frac{1}{N} \cdot \overline{\underline{\underline{F}}} \cdot \underline{X} \qquad \text{Frequency} \rightarrow \textit{Time} \tag{5.30}$$

Matrix $\overline{\underline{\underline{F}}}$ is the **Hermitian adjoint** of $\underline{\underline{F}}$ (Chapter 2). It follows from Equations 5.27 and 5.30 that $\underline{x} = 1/N \cdot \overline{\underline{\underline{F}}} \cdot \left(\underline{\underline{F}} \cdot \underline{x}\right)$. Then

$$\underline{\underline{I}} = \frac{1}{N} \cdot \overline{\underline{\underline{F}}} \cdot \underline{\underline{F}} \tag{5.31}$$

Implementation Procedure 5.1 outlines the implementation of Fourier transform operations in matrix form.

Implementation Procedure 5.1 Fourier analysis in matrix form

1. Digitize the signal $x(t)$ with a sampling interval At to generate the array $\underline{x}$ [N x 1].
2. Create the Fourier transformation matrix $\underline{\underline{F}}$:

$$F_{u,i} = e^{-j\cdot\left(u\frac{2\pi}{N}i\right)}$$

   for i and u that range between [0, . . N − 1]. The matrix is symmetric.
3. The DFT of the signal $\underline{x}$ is $\underline{X} = \underline{\underline{F}} \cdot \underline{x}$.
4. The magnitude and the phase of each frequency component are

$$\text{Magnitude: } |X_u| = \sqrt{[\mathrm{Re}(X_u)]^2 + [\mathrm{Im}(X_u)]^2}$$

$$\text{Phase: } \varphi_u = \tan^{-1}\left[\frac{\mathrm{Im}(X_u)}{\mathrm{Re}(X_u)}\right]$$

   for corresponding frequency: $f_u = u\frac{1}{N \cdot At}$ or $\omega_u = u\frac{2\pi}{N \cdot At}$

5. Conversely, given a signal in the frequency domain $\underline{X}$, its IDFT is the time domain signal $\underline{x}$,

$$\underline{x} = \frac{1}{N}\overline{\underline{\underline{F}}} \cdot \underline{X} \quad \text{where} \quad \overline{F_{u,i}} = \text{complex conjugate of } F_{u,i}$$

Note: The fast Fourier transform (FFT) is preferred for large signals. The FFT algorithm is included in all commercially available mathematical software and in public domain codes at numerous internet sites.

5.5 TRUNCATION, LEAKAGE, AND WINDOWS

Short duration transients can be adequately recorded from beginning to end. Some A/D converters even permit pretriggering to gather the background signal prior to the transient. However, long-duration or ongoing signals are inevitably truncated and we only see a finite "window of the signal".

The effects of truncation are studied with a numerical example in Figure 5.6. The sinusoid is truncated when six cycles are completed (Figure 5.6a). The autospectral density is shown in Figure 5.6b. Given that this is a single-frequency sinusoid, the autospectral density is an impulse at the frequency of the signal. Figure 5.6c shows a similar signal truncated after 5.5 cycles. The autospectral density is shown in Figure 5.6d. In contrast to the previous case, energy has "leaked" into other frequencies.

Leakage is the consequence of two inherent characteristics of the DFT. The first one is the ***unmatched harmonic*** effect whereby the sinusoid frequency f^* in Figure 5.6c is not a harmonic of $f_{min} = 1/(N \cdot \Delta t)$; therefore, the DFT cannot produce an impulse at f^*. Instead, the DFT "curve-fits" the signal with harmonically related sinusoids at frequencies $f_u = u/(N \cdot \Delta t)$. The second cause for leakage results from the ***presumed periodicity*** in the DFT: the signal in Figure 5.6c is effectively considered the periodic signal in Figure 5.6e. The resulting sharp discontinuities at the end of the signal require higher-frequency components; in addition, the lack of complete cycles leads to a nonzero static component.

The window imposed on the analog signal during A/D conversion into a finite record is a sharp-edged off–on–off window and magnifies discontinuity effects. Leakage is reduced by "windowing the signal" with gradual window arrays $\underline{w}$. The windowed signal $\underline{x}^{<win>}$ is obtained multiplying the signal $\underline{x}$ with the window $\underline{w}$ point by point:

$$x_i^{<win>} = x_i \cdot w_i \tag{5.32}$$

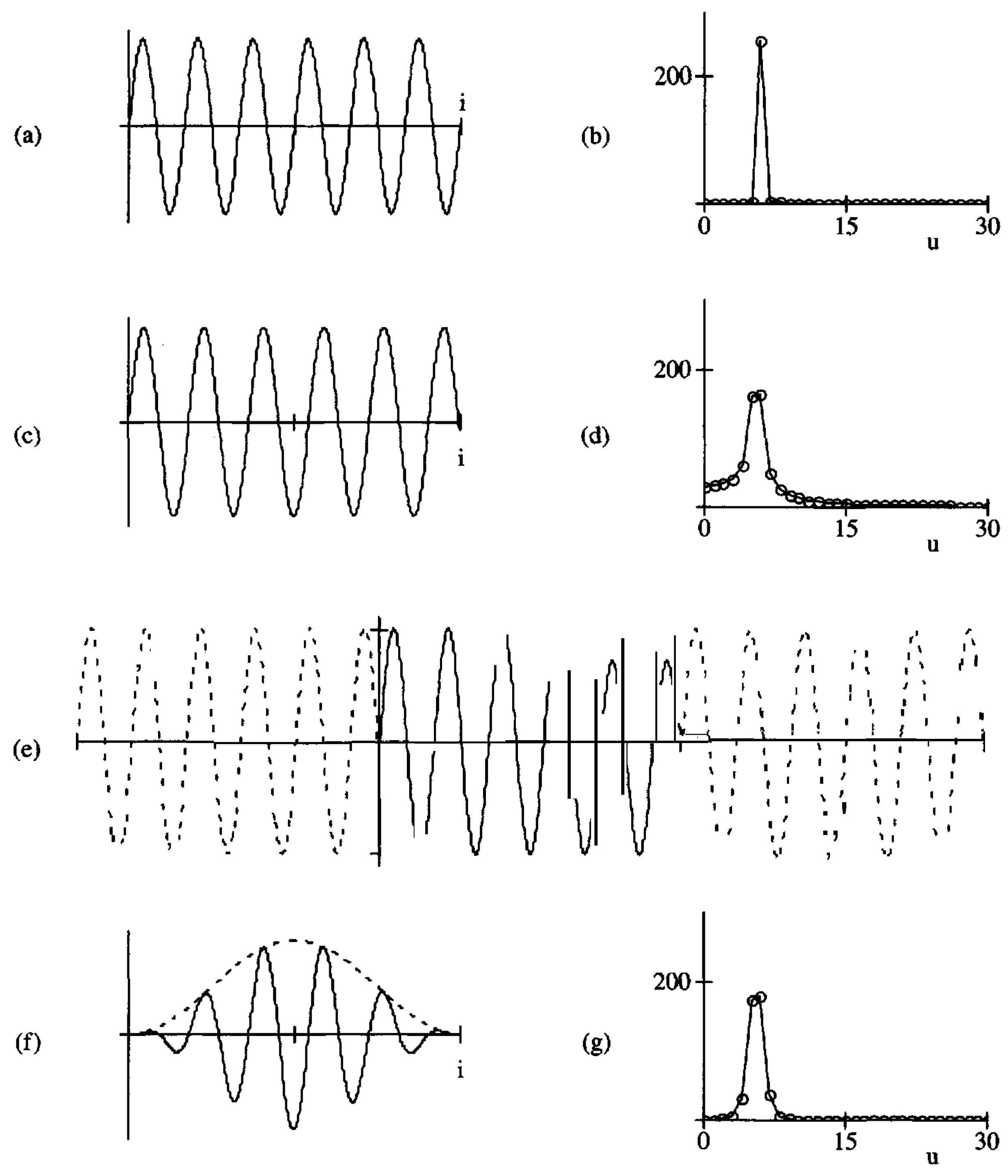


Figure 5.6 Truncation and windowing: (a, b) the DFT of a single frequency sinusoid is an impulse if it completes an integer number of cycles in the duration of the signal T; (c, d) this signal has an incomplete number of cycles; its DFT is not an impulse and has a static component; (e) periodic assumption in the **DFT**; (f) signal in frame 'c' but windowed with smooth transition towards zero ends; (g) autospectrum of the windowed signal

The Hanning and Hamming windows are two common windowing functions:

$$\textit{Hanning} \qquad w_i = \begin{cases} \frac{1}{2} + \frac{1}{2} \cdot \cos\left[\frac{2\pi}{E}(i-M)\right] & |i-M| \le \frac{E}{2} \\ 0 & \text{otherwise} \end{cases} \tag{5.33}$$

$$\textit{Hamming} \qquad w_i = \begin{cases} 0.54 + 0.46 \cdot \cos\left[\frac{2\pi}{E}(i-M)\right] & |i-M| \le \frac{E}{2} \\ 0 & \text{otherwise} \end{cases} \tag{5.34}$$

These windows are centered around i = M and have a time width E . At. In this format, the rectangular window becomes

$$\textit{Rectangular} \; w_i = \begin{cases} 1 & |i-M| \le \frac{E}{2} \\ 0 & \text{otherwise} \end{cases} \tag{5.35}$$

Figure **5.6f** shows the signal in Figure **5.6c** when the Hanning window is used. Finally, Figure **5.6g** shows the autospectral density of the windowed signal.

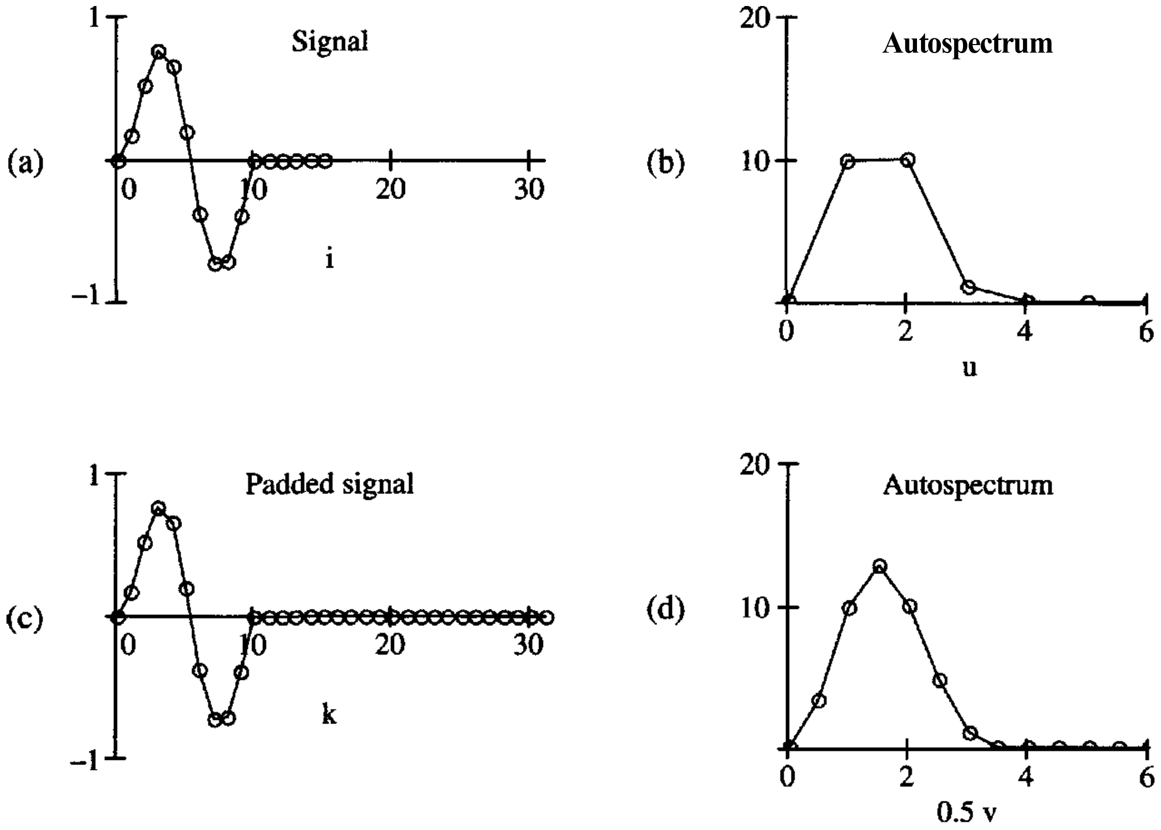
The energy available in the windowed signal is reduced by windowing. The ratio of the energy in the original signal $\underline{x}$ and the windowed signal $\underline{x}^{<win>}$ can be computed in the time domain:

$$\beta = \sqrt{\frac{\sum_{i=0}^{N-1} x_i^2}{\sum_{i=0}^{N-1} \left(x_i^{<win>}\right)^2}} \tag{5.36}$$

5.6 PADDING

A longer duration N · At signal renders a better frequency resolution Af = 1/(N · At). Therefore, a frequently used technique to enhance the frequency resolution of a stored signal length N consists of "extending" the signal by appending values to a length M > N. This approach requires careful consideration.

There are various "signal extension" strategies. Zero padding, the most common extension strategy, consists of appending zeros to the signal. Constant padding extends the signal by repeating the last value. Linear padding extends the signal while maintaining the first derivative at the end of the signal constant. Finally, periodic padding uses the same signature for padding.



F i r e 5.7 Time and frequency resolution: (a, b) original $N = 16$ signal and its auto spectrum; (c, d) zero-padded signal with $N = 32$ and its auto spectrum. Padding increases frequency resolution. The peak in the autospectral density of the original signal is absent because there is no corresponding harmonic. (Note: the time interval At is kept constant, the number of points N is doubled, and the frequency interval is halved.)

Figure 5.7 presents an example of zero padding. The signal length is $N = 16$ and the DFT decomposes it into harmonics $f_u = u/(16\Delta t)$, while the padded signal is length $M = 32$ and the associated harmonics are $f_v = v/(32\Delta t)$. The sinusoid duration is $11 \cdot \Delta t$; thus, its main frequency is $f^* = 1/(11 \,.\, At)$. Therefore, the harmonic for $v = 3$ in the DFT of the padded signal is quite close to f^*, but there is no harmonic in the DFT of the original signal near f^*.

The following observations follow from this example and related analyses:

Signal extension is not intended to add information. Therefore, there is no new information in the frequency domain if the same information is available in the time domain.

- The real effect of padding is to create harmonic components that better “fit” the signal.
- Zero and periodic padding may create discontinuities; plot the signal in the time domain to verify continuity.

- The negative effects of padding are reduced when signals are properly detrended and windowed first.
- The signal length can be increased by adding zeros at the front of the signal; however, this implies a time shift in all frequencies, and a frequency-dependent phase shift, as predicted in Equation 5.22.
- Signal extension to attain a signal length $N = 2^r$ allows the use of more computationally efficient Fast Fourier transform algorithms. However, harmonics may be lost: for example, a sinusoid with period $450 \cdot \Delta t$ in a signal length $N = 900$ has a harmonic at $u = 2$, but it has no harmonic when the signal is zero padded to $M = 2^{10} = 1024$.
- When the main frequency in the signal under study is a known value f^*, then record length N and sample interval Δt are selected so that f^* is one the harmonics $f_u = u/(N\Delta t)$ in the discrete spectrum.

  The DFT presumes the signal is periodic with fundamental period $T = N \cdot \Delta t$. Signal extension increases the fundamental period and prevents circular convolution effects in frequency domain computations (Chapter 6).
- The previous observations apply to deterministic signals. In the case of random signals, signal extension must preserve stationary conditions.

Enhanced resolution with harmonics that better "fit" the signal lead to more accurate system identification (review Figure 5.7). Consider a low-damping single degree of freedom oscillator: the narrow resonant peak may be missed when the frequency resolution is low and no harmonic f_u matches the resonant frequency. In this case, the inferred natural frequency and damping of the oscillator would be incorrect.

5.7 PLOTS

A signal in the time domain (time or space) is primarily plotted as x_i versus time $t_i = i \cdot \Delta t$. However, there are several alternatives in the frequency domain to facilitate the interpretation of the information encoded in the signal. Consider the signal in Figure 5.8a, which shows the free vibration of an oscillator after being excited by a very short impulse-like input signal. Various plots of the DFT are shown in Figures 8b–h:

- Figure 5.8b shows the autospectral density versus the frequency index u. The first mode of vibration is clearly seen. When the autospectral density is plotted in log scale, other low-amplitude vibration modes are identified (Figure 5.8c).

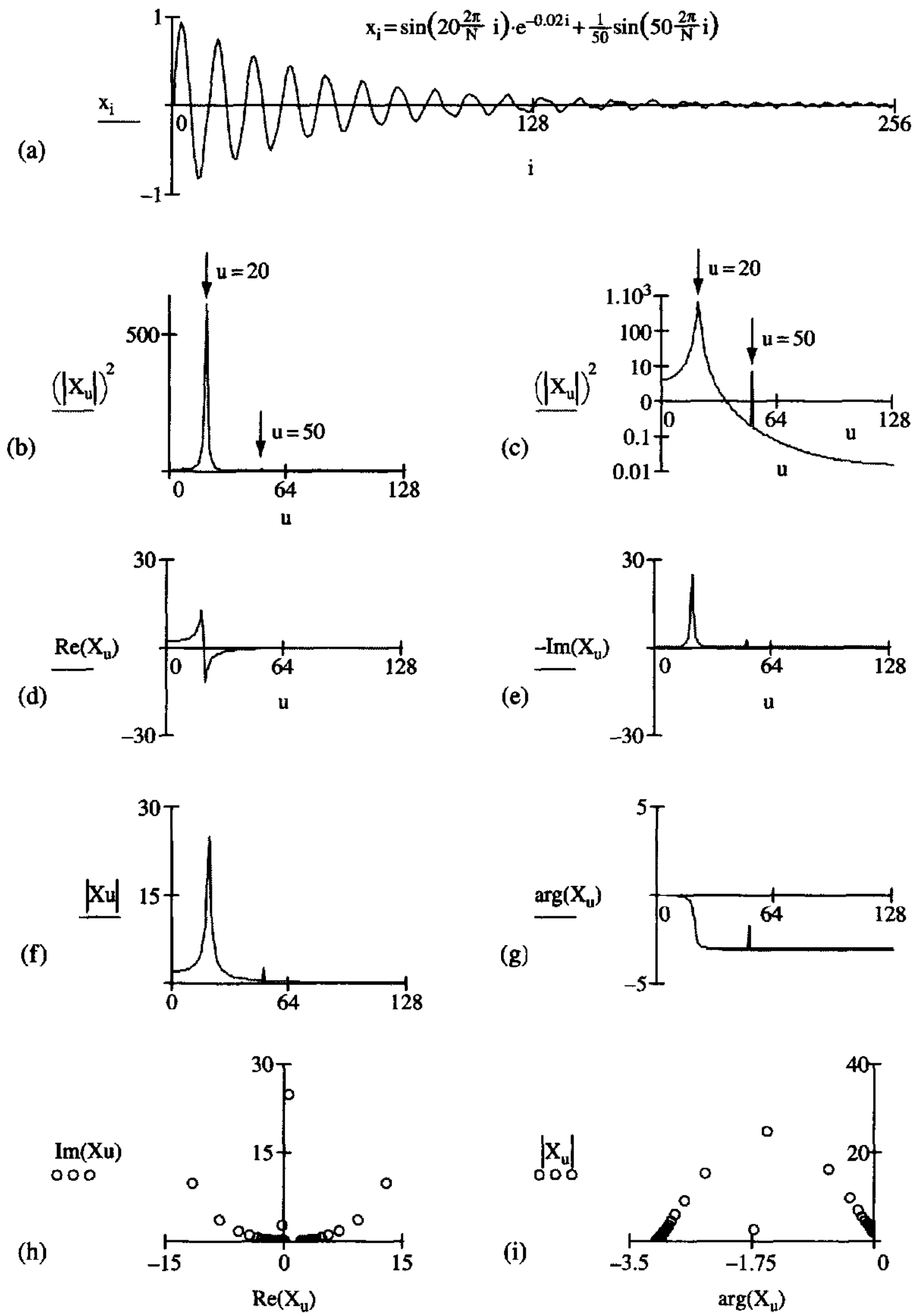


Figure 5 8 Different plots of the DFT of a signal: (a) original signal $\underline{x}$ in time domain; (b, c) autospectral density – normal and log magnitudes; (d, e) real and imaginary components versus frequency index u; (f, g) amplitude and phase versus frequency index u; (h) imaginary versus real component (Cole-Cole plot); (i) amplitude versus phase. Frequency domain data are presented single-sided, for u = [0, N/2]

- Figures 5.8d and e show the real $Re(X_u)$ and imaginary $Im(X_u)$ components of the DFT versus the frequency index u.
- Figures 5.8f and g show the amplitude $|X_u|$ and the phase φ_u versus the frequency index u.
- Figure 5.8h shows the imaginary component $Im(X_u)$ versus the real component $Re(X_u)$. This is called the Cole-Cole plot, and it is used to identify materials that show relaxation behavior (e.g. response of a viscoelastic material); a relaxation defines a semicircle in these coordinates.
- Figure 5.8i shows a plot of amplitude versus phase.

Any frequency is readily recovered from the frequency counter u as $f_u = u/(N\Delta t)$. In particular, the frequency associated with the peak response is the oscillator resonant frequency. The oscillator damping is reflected in both time and frequency domains: low damping is denoted by multiple oscillations in the time domain (Figure 5.8) and a narrow peak in the frequency domain (Chapter 4).

5.8 THE TWO-DIMENSIONAL DISCRETE FOURIER TRANSFORM

A 2D signal $x(p, q)$ captures the variation of a parameter in two dimensions p and q. During A/D conversion, the signal is digitized along a grid made of M discrete values in p and N discrete values in q. The discrete 2D signal is a matrix $\underline{\underline{x}}$ where entry $x_{i,k}$ corresponds to location $p = i \cdot Ap$ and $q = k . Aq$. The 2D signal may involve data gathered in any two independent dimensions, such as a digital picture or a sequence of time series obtained at different positions in space.

The DFT $\underline{\underline{X}}$ of $\underline{\underline{x}}$ is also a matrix; each entry $X_{u,v}$ corresponds to frequencies $f_u = u/(M \cdot \Delta p)$ and $f_v = v/(N . Aq)$. The 2D Fourier transform pair is

$$X_{u,v} = \sum_{i=0}^{M-1} \left[\sum_{k=0}^{N-1} x_{i,k} \cdot e^{-j\left(v \cdot \frac{2\pi}{N} \cdot k\right)} \right] \cdot e^{-j\left(u \cdot \frac{2\pi}{M} \cdot i\right)} \qquad \text{2D } \textit{Analysis} \qquad (5.37)$$

$$x_{i,k} = \frac{1}{M} \cdot \sum_{u=0}^{M-1} \left[\frac{1}{N} \cdot \sum_{v=0}^{N-1} X_{u,v} \cdot e^{j\left(v \cdot \frac{2\pi}{N} \cdot k\right)} \right] \cdot e^{j\left(u \cdot \frac{2\pi}{M} \cdot i\right)} \qquad \text{2D } \textit{Synthesis} \qquad (5.38)$$

The 2D DFT can be computed with 1D algorithms in two steps. First, an intermediate matrix $\underline{\underline{INT}}$ is constructed where each row is the DFT of the corresponding row of $\underline{\underline{x}}$. The columns of the final 2D Fourier transform $\underline{\underline{X}}$ are obtained by computing the DFT of the corresponding columns in $\underline{\underline{INT}}$.

Analysis and synthesis operations can be expressed in matrix form, in analogy to the case of 1D signals. In particular, if the discrete signal is square M = N, the 2D Fourier transform of $\underline{\underline{x}}$ is

$$\underline{\underline{X}} = \left[\underline{\underline{F}} \cdot \left(\underline{\underline{F}} \cdot \underline{\underline{x}}\right)^T\right]^T = \underline{\underline{F}} \cdot \underline{\underline{x}} \cdot \underline{\underline{F}} \quad \textit{from } p\text{–}q \textit{ to } f_p - f_q \qquad (5.39)$$

where the second equality follows from $\underline{\underline{F}}^T = \underline{\underline{F}}$. The k-th element in the v-th row of $\underline{\underline{F}}$(N x N) is

$$F_{v,k} = e^{-j \cdot \left(v \cdot \frac{2\pi}{N} \cdot k\right)} \qquad (5.40)$$

Because $N \cdot \underline{\underline{I}} = \overline{\underline{\underline{F}}} \cdot \underline{\underline{F}}$ (Equation 5.31), the synthesis equation in matrix form is

$$\underline{\underline{x}} = \frac{1}{N^2} \cdot \overline{\underline{\underline{F}}} \cdot \underline{\underline{X}} \cdot \overline{\underline{\underline{F}}} \qquad \textit{from } f_p - f_q \textit{ to } p - q \qquad (5.41)$$

Other concepts such as resolution, truncation and leakage, discussed in relation to 1D signals, apply to 2D signals as well.

Examples of 2D DFT are presented in Figure 5.9 (see solved example at the end of this Chapter). The following observations can be made (analogous to the 1D DFT Figure 5.1). The DFT of a uniform 2D signal has only the real DC component at u = 0, v = 0 (Figure 5.9a). The DFT of the linear combination of 2D signals is the linear combination of DFT of the individual signals (Figure 5.9b). A single frequency sinusoid becomes an impulse in the frequency domain in the same direction as the signal in the time domain (Figure 5.9c); if there is leakage, it manifests parallel to the u and v axes.

5.9 PROCEDURE FOR SIGNAL RECORDING

The most robust approach to signal processing is to ***improve the data at the lowest possible level*** (review Section 4.1.5). Start with a proper experimental design: explore various testing approaches, select the transducers that are best fitted to sense the needed parameter under study, match impedances, reduce noise by proper insulation (electromagnetic, mechanical, thermal, chemical, and biological) and use quality peripheral electronics. If the signal is still poor, then the option of signal stacking should be considered before analog filters are included in the circuitry.

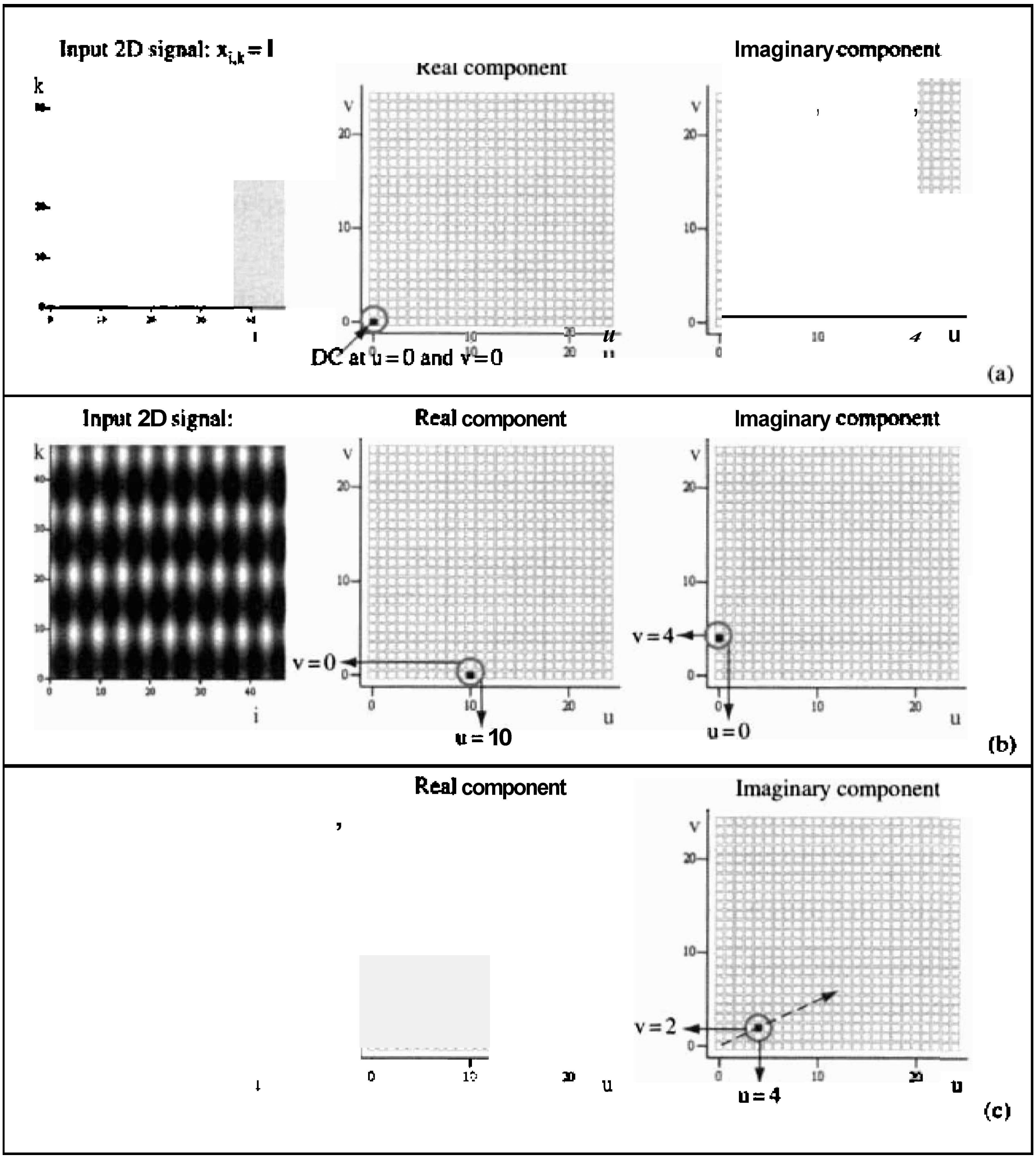


Figure 5.9 The 2D-DFT: (a) constant-value signal: the only nonzero value in 2D-DFT is the DC component; (b) the signal $x_{i,k} = \cos(10\frac{2\pi}{N}i) + \sin(4\frac{2\pi}{N}i)$ has one peak in the real part and one peak in the imaginary components of the 2D-DFT–note the direction in each case relative to the image; (c) the 2D-DFT of the single frequency sinusoid $x_{i,k} = \sin\left[4\frac{2\pi}{N}(i+0.5k)\right]$ is aligned in the same direction as the oscillations in the signal

Once these recommendations have been taken into consideration, start planning the signal digitization and storage. Concepts discussed in this and the previous chapters permit outlining of common guidelines for signal recording that are applicable to most situations. When signal processing involves DFTs, data gathering must consider signal length, truncation and leakage, windowing, and frequency resolution. Guidelines are summarized in the Implementation Procedure 5.2.

Implementation Procedure 5.2 Recommended procedure for signal recording

1. The signal must be improved at the lowest possible level, starting with a carefully designed experimental setup, adequate choice of electronics, and proper isolation of the system to reduce noise.
2. It is advantageous to extend the recording duration T so that zero amplitude is recorded at the front and tail ends of the signal. This is possible in short-duration events.
3. The sampling interval or time resolution At must be selected to properly digitize the highest-frequency component of interest f_{max}, fulfilling the Nyquist criterion. It is recommended that At $\sim 1/(10 \cdot f_{max})$ be used. If unwanted higher frequency components are expected, they should be removed with an analog filter before digitalization. Many A/D systems include antialiasing filters at the input to automatically remove frequency components that would be aliased otherwise.
4. The total number of points to be recorded is estimated as $N = T/\Delta t$. If you know the main frequency in the signal under study f*, then combine record length N and sample interval At so that f* is one of the harmonics $f_u = u/(N\Delta t)$ in the discrete spectrum.
5. Detrend and remove spikes in the signal before the signal is transformed.
6. Window truncated signals to reduce leakage. Windowing and zero-offset corrections may be repeated.
7. Extend the recorded signal to increase frequency resolution. Make sure that there is a harmonic in the padded signal that corresponds to the main component f* in the signal under study.

5.10 SUMMARY

- Harmonically related sinusoids and complex exponentials are orthogonal functions in the open interval [0, T[. Therefore, they form a base that can be used to express any other function as a linear combination. This is the foundation for the DFT.
- For a robust interpretation of the DFT of a signal length N, remember that: (1) the DFT is equivalent to fitting the signal with a series of cosines and sines and storing the amplitudes in the "real" and "imaginary" arrays, (2) the signal

is assumed periodic with period equal to the duration of the signal $T = N \cdot \Delta t$, and (3) only harmonically related sinusoid frequencies $f_u = u/(N \cdot \Delta t)$ are used.

- The DFT has a finite number of terms. In fact, there are N information units in a signal length N, both in the time domain and in the frequency domain. There are no convergence difficulties in the DFT of discrete time signals.
- The parallelism between analysis and synthesis relations in a Fourier pair leads to the duality of the DFT.
- Resolution in time is inversely proportional to resolution in frequency. Signal extension or padding decreases the frequency interval between consecutive harmonics.
- The truncation of ongoing signals produces leakage. Leakage effects are reduced by windowing signals with smooth boundary windows.
- The DFT can be applied to signals that vary along more than one independent variable, such as 2D images or data in space-time coordinates.

The signal must be improved at the lowest possible level, starting with careful experimental setup, adequate choice of electronics, and proper isolation of the system under study to reduce noise. While planning analog-to-digital conversion, the experimenter must take into consideration the system under study and the mathematical implications of digitization and DFT operations.

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SOLVED PROBLEMS

P5.1 Fourier **series.** Demonstrate that:

$$\int_0^T e^{j\left(\frac{2\pi}{T}t\right)} \cdot e^{-j\left(u\frac{2\pi}{T}t\right)} \cdot dt = \begin{cases} 0 & \text{if } u \neq 1 \\ T & \text{if } u = 1 \end{cases}$$

Solution: Using Euler's identities

$$f(T) = \int_0^T \left[\cos\left(\frac{2\pi}{T}t\right) + j \cdot \sin\left(\frac{2\pi}{T}t\right)\right] \cdot \left[\cos\left(u\frac{2\pi}{T}t\right) - j \cdot \sin\left(u\frac{2\pi}{T}t\right)\right] \cdot dt$$

$$f(T) = \int_0^T \begin{bmatrix} \cos\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right) - j \cdot \cos\left(\frac{2\pi}{T}t\right) \cdot \sin\left(u\frac{2\pi}{T}t\right) \\ +j \cdot \sin\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right) + \sin\left(\frac{2\pi}{T}t\right) \cdot \sin\left(u\frac{2\pi}{T}t\right) \end{bmatrix} \cdot dt$$

$$f(T) = \int_0^T \left[\cos\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right)\right] \cdot dt - j \cdot \int_0^T \left[\cos\left(\frac{2\pi}{T}t\right) \cdot \sin\left(u\frac{2\pi}{T}t\right)\right] \cdot dt$$

$$+j \cdot \int_0^T \left[\sin\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right)\right] \cdot dt + \int_0^T \left[\sin\left(\frac{2\pi}{T}t\right) \cdot \sin\left(u\frac{2\pi}{T}t\right)\right] \cdot dt$$

Invoking Equation 5.4, the previous equation simplifies to

$$f(T) = \int_0^T \left[\cos\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right)\right] \cdot dt + \int_0^T \left[\sin\left(\frac{2\pi}{T}t\right) \cdot \sin\left(u\frac{2\pi}{T}t\right)\right] \cdot dt$$

And, from Equations 5.3 and 5.4:

$$f(T) = \underbrace{\int_0^T \left[\cos\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right)\right] \cdot dt}_{\begin{matrix} 0 & \text{if} & u \neq 0 \\ \frac{T}{2} & \text{if} & u \neq 0 \end{matrix}} + \underbrace{\int_0^T \left[\sin\left(\frac{2\pi}{T}t\right) \cdot \sin\left(u\frac{2\pi}{T}t\right)\right] \cdot dt}_{\begin{matrix} 0 & \text{if} & u \neq 0 \\ \frac{T}{2} & \text{if} & u \neq 0 \end{matrix}}$$

P5.2 ***Digitization.*** Given a sampling interval $At = 10^{-3}$s and a record length $T = 0.5$s, compute: (a) frequency resolution, (b) frequency corresponding to the frequency counter $u = 13$, (c) the shortest time shift compatible with a phase shift $\Delta\phi = \pi$ for the frequency component that corresponds to $u = 10$.

Solution:

(a) The frequency resolution is $\Delta f = \frac{1}{T} = \frac{1}{0.5s} = 2Hz$

(b) The frequency corresponding to u = 13 is $f_{13} = u \cdot \Delta f = f = 13 \cdot 2\,Hz = 26\,Hz$

(c) Phase and time shifts are related as $\frac{u}{2\pi} = \frac{\delta t}{T_u}$

The time shift is $\delta t = \frac{\Delta\phi_u}{2\pi}\frac{T}{u} = 0.025\,s$

P5.3 ***2D-Fourier transform.*** Create a 2D image $\underline{\underline{x}}$ to represent ripples on a pond. Calculate the discrete Fourier transform $\underline{\underline{X}}$. Analyze the results.

Solution: Definition of function $\underline{\underline{x}}$ ($N \times N$ elements where $N = 64$)

Distance from the center of the pond: $r_{i,k} = \sqrt{\left(i - \frac{N}{2}\right)^2 + \left(k - \frac{N}{2}\right)^2}$

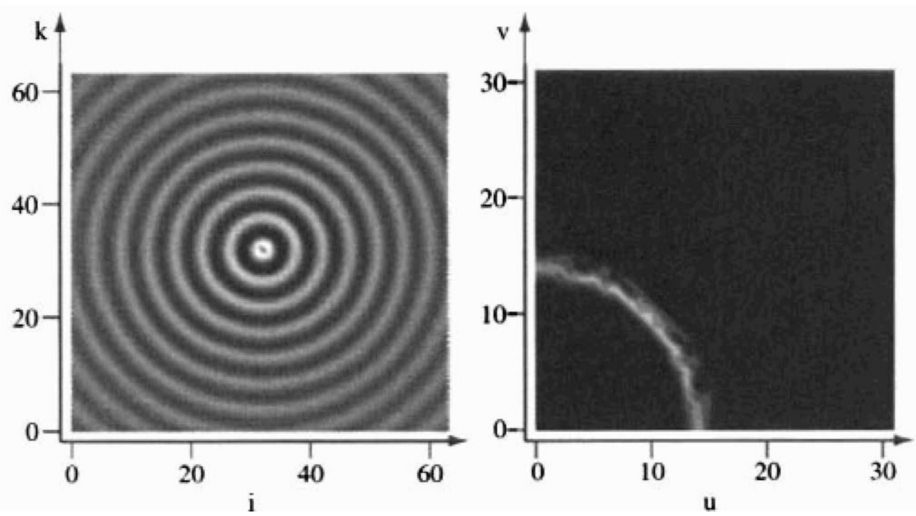
Displacement function: $x_{i,k} = \frac{\sin\left(10\frac{2\pi}{\max(r)} r_{i,k}\right)}{r_{i,k} + 10}$

Discrete Fourier transform matrix: $F_{u,i} = e^{-ju\frac{2\pi}{N}i}$

2D discrete Fourier transform: $\underline{\underline{X}} = \underline{\underline{F}} \cdot \underline{\underline{x}} \cdot \underline{\underline{F}}$

Magnitude: $|X_{u,v}| = X_{u,v} \cdot \overline{X_{u,v}}$

where the spatial indices i and k range from 0 to $N - 1$ and the frequency indices u and v range from 0 to $N - 1$. Time and frequency domain plots are presented next. Only one quadrant of the 2D-DFT is shown:



Interpretation: There are 15 ripples in both i and k directions along the center of the plot. That is the location of peak energy along the u and v axis.

Explore this solution further. What happens if you shift the center of the ripples away from the center of the image? What is the 2D-DFT of a signal with elliptical ripples?

ADDITIONAL PROBLEMS

P5.4 ***Fourier series.*** Compute the fourth-order discrete Fourier series ($u = 0, 1, 2, 3, 4$) that best approximates an odd square wave. Repeat for an even square wave. Compare the coefficients for sine and cosine components in both cases. What can be concluded about the decomposition of even and odd signals?

P5.5 ***Discrete Fourier transform pairs.*** There are various Fourier pairs besides the one presented in Table 5.1; for example:

Analysis: $a_u = \frac{1}{N}\sum_{i=0}^{N-1} x_i \cdot \cos\left(u\frac{2\pi}{N}i\right)$ and $b_u = \frac{1}{N}\sum_{i=0}^{N-1} x_i \cdot \sin\left(u\frac{2\pi}{N}i\right)$

Synthesis: $x_i = \sum_{u=0}^{N-1}\left[a_u \cdot \cos\left(u\frac{2\pi}{N}i\right) + j\cdot b_u \cdot \sin\left(u\frac{2\pi}{N}i\right)\right]$

Determine the relationship between this Fourier pair and the one presented in Table 5.1. Explicitly state the relationship between a_u and b_u and X_u.

P5.6 ***Properties of the discrete Fourier transform.*** Demonstrate the following properties of the DFT of discrete periodic signals: linearity, periodicity, differentiation, Parseval's relation, time shift, and $N\cdot\underline{\underline{I}} = \overline{\underline{\underline{F}}}\cdot\underline{\underline{F}}$ (matrix operations). Is the magnification of high-frequency components linear with frequency in Equation 5.23?

P5.7 ***Single-sided discrete Fourier transform.*** Use the properties of the DFT to show that the computation of the DFT can be reduced to coefficients $u = 0$ to $u = N/2$. Rewrite the synthesis equation to show this reduced summation limits. Corroborate your results using numerical simulation. Compare the autospectral density in both cases.

P5.8 ***Discrete Fourier transform of a complex exponential.*** What is the DFT of a complex exponential? Consider both positive and negative exponents. Solve this problem both analytically and numerically. (Important: use double sided formulation, that is, from $u = 0$ to $N - 1$; this exercise is revisited in Chapter 7.)

P5.9 ***Padding.*** Generate a $N = 300$ points sinusoid $x_i = \sin(8 \cdot \frac{2\pi}{N} \cdot i)$. Consider different padding criteria to extend the signal to N = 512 points and compute the **DFT** in each case. Analyze spectra in detail and draw conclusions.

P5.10 ***Application: signal recording and preprocessing.*** Capture a set of signals within the context of your research interests. Follow the recommendations outlined in the Implementation Procedure 5.3. For each signal:

- **Detrend** the signal.
- Window the signal with a Hamming window (test different widths E).
- Compute the DFT and plot results in different forms to highlight the underlying physical process.
- Infer the characteristics of the system (e.g. damping and resonance if testing a single **DoF** system).
- Double the number of points by padding, compute the **DFT** and compare the spectra with the original signals.
- Repeat the exercise varying parameters such as sampling interval At, number of stored points N, and signal amplitude.

P5.11 ***Application: sound*** and ***octave analysis.*** The ***octave*** of a signal frequency f is the first harmonic 2f. In "octave analysis", frequency is plotted in logarithmic scale. Therefore, the central frequency of each band increases logarithmically, and bins have constant log-frequency width; that is, the frequency width of each bin increases proportionally to the central frequency. Systems that operate with octave analysis include filters with upper-to-lower frequency ratio 2n, where n is either 1, 112, 116, or 1112. This type of analysis is preferred in studies of sound and hearing. Create a frequency sweep sinusoid $\underline{x}$ with frequency increasing linearly with time. Plot the signal. Compute $\underline{X} = \text{DFT}(\underline{x})$. and plot the magnitude versus linear and logarithmic frequency. Draw conclusions.

P5.12 ***Application: Walsh series.*** A signal can be expressed as a sum of square signals with amplitude that ranges between +1 and –1. In particular, the Walsh series is orthogonal, normalized, and complete. Research the Walsh series and:

1. Write the Walsh series in **matrix** form (length N = 16).
2. Study the properties of the matrix. Is it invertible?

3. **Apply the Walsh's decomposition to a sinusoidal signal, a stepped signal (e.g. transducer with digital output), and to a small digital image.**
4. **Analyze your results and compare with Fourier approaches (see also the Hadamard transform).**