Engineering Seismology and Seismic Hazard – 2019

Lecture 5

Basic Math Review

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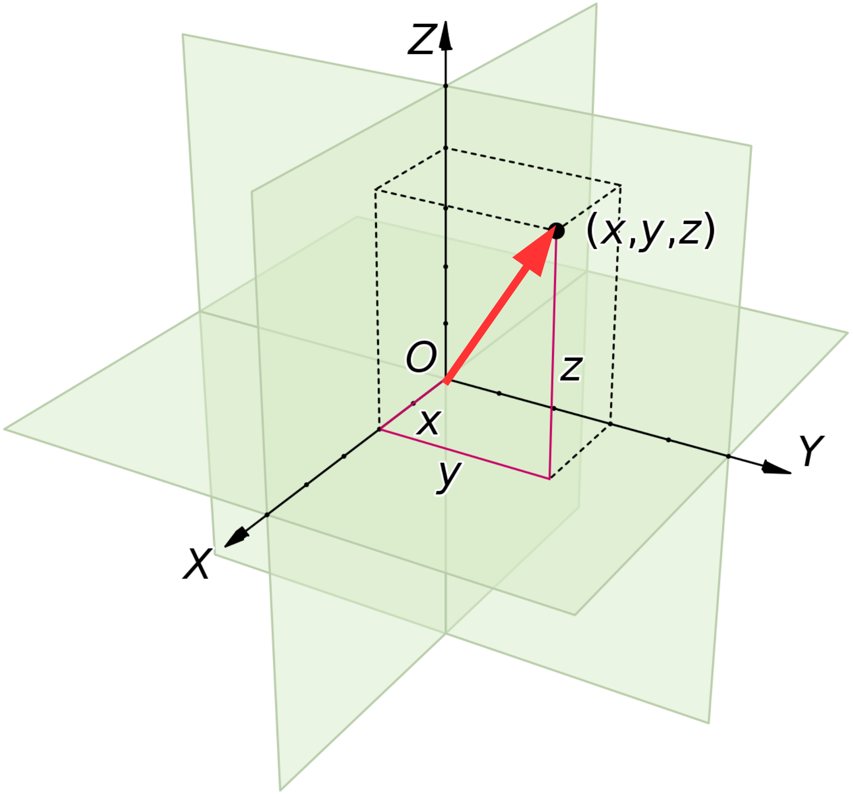
Vectors

In the following lectures, we will heavily rely on the concept of vectors and tensors.

A vector is an object that has both a magnitude and a direction

A vector of dimension *n* is an ordered collection of *n* (scalar) elements, called components.

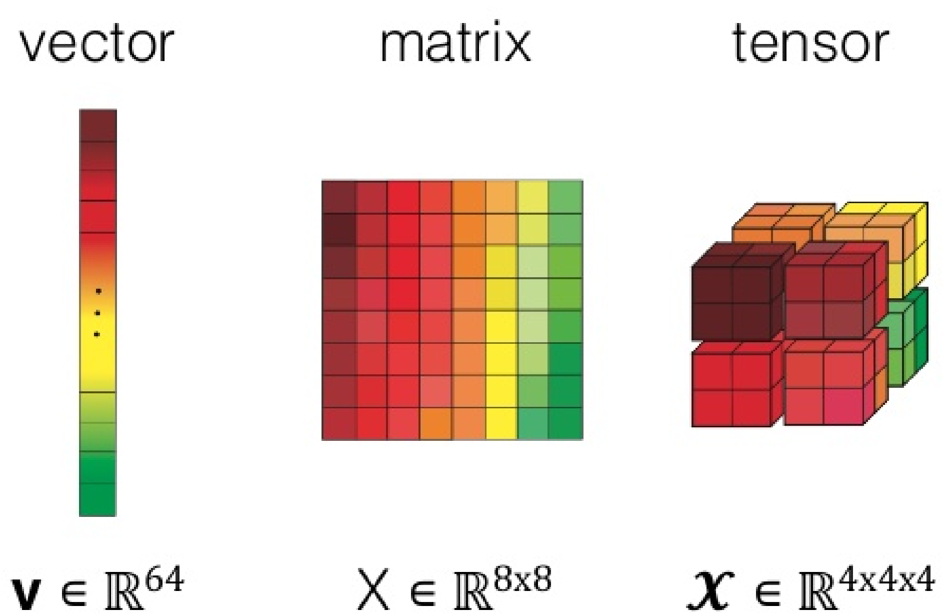
$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$



Tensors

Tensors are mathematical objects that represent the generalizations of scalars (that have no indices), vectors (that have exactly one index), and matrices (that have exactly two indices) to an arbitrary number of indices n

tensor = multidimensional array

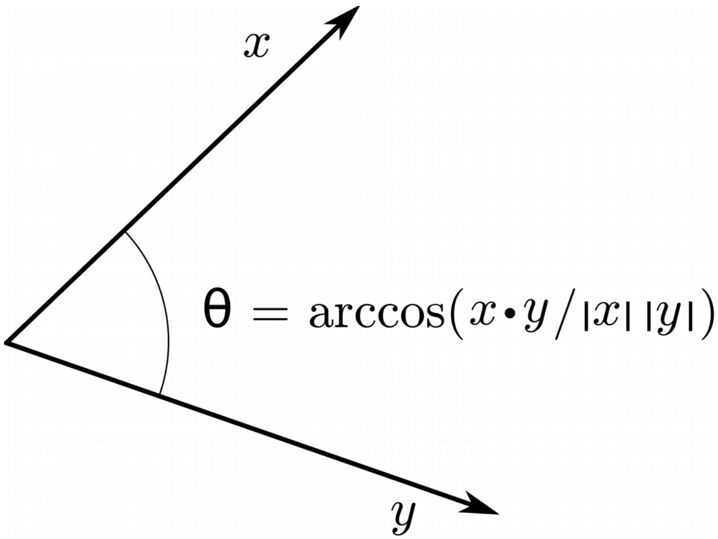


$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Scalar or Dot Product

The dot product, also called the scalar product, of two vector is a number (scalar quantity) obtained by performing the following operation:

$$\vec{a}\cdot\vec{b}=\sum_{i=1}^{n} a_i b_i$$



In 3-dimensional Euclidean space:

$$\vec{a}\cdot\vec{b}=a_1 b_1+a_2 b_2+a_3 b_3=|\vec{a}||\vec{b}|\cos(\theta)$$

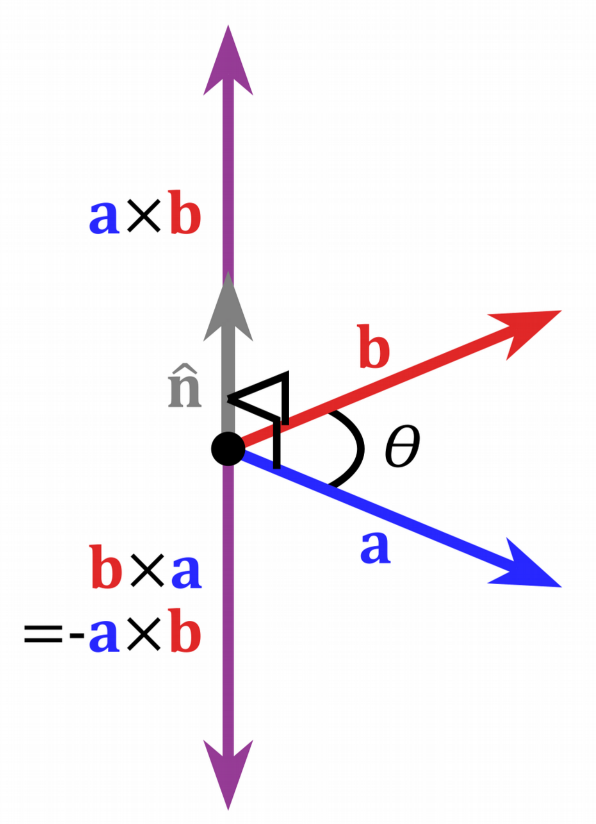
Vector or Cross Product

The cross product or vector product is a binary operation on two vectors in three-dimensional space, given by:

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\hat{i} + (a_3 b_1 - a_1 b_3)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

And geometrically:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\theta)$$



Triple Product

Scalar triple product:

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b})$$

Vector triple product:

$$\vec{a}\times(\vec{b}\times\vec{c})=b(\vec{a}\cdot\vec{c})-c(\vec{a}\cdot\vec{b})$$

Gradient Operator

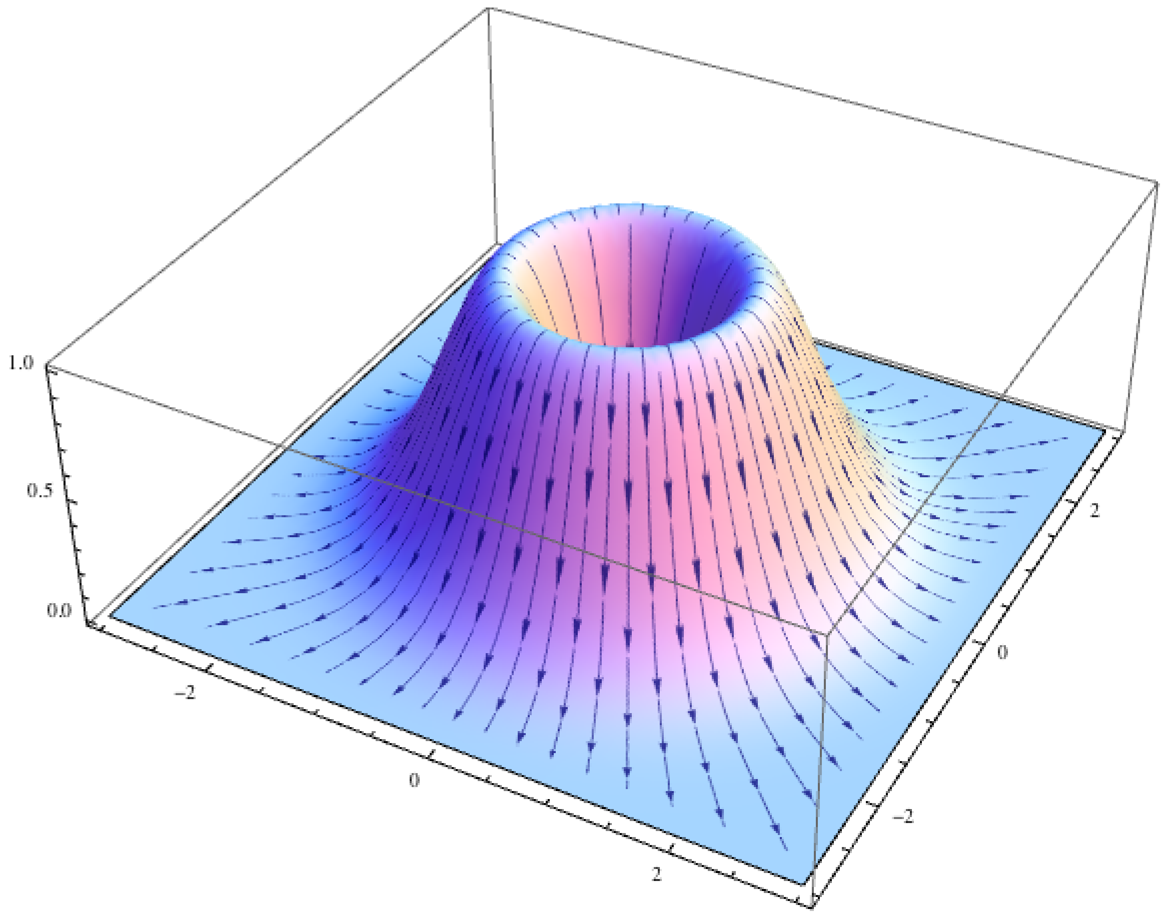
The gradient operator is a vector containing three partial derivatives. When applied to a scalar, it produces a vector, when applied to a vector, it produces a tensor.

$$\nabla a = \left[\frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}, \frac{\partial a}{\partial z} \right]$$

$$\nabla \vec{b} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [b_x, b_y, b_z] = \begin{bmatrix} \frac{\partial b_x}{\partial x} & \frac{\partial b_y}{\partial x} & \frac{\partial b_z}{\partial x} \\ \frac{\partial b_x}{\partial y} & \frac{\partial b_y}{\partial y} & \frac{\partial b_z}{\partial y} \\ \frac{\partial b_x}{\partial z} & \frac{\partial b_y}{\partial z} & \frac{\partial b_z}{\partial z} \end{bmatrix}$$

Gradient Operator

The gradient vector of a scalar quantity defines the direction in which it increases fastest; the magnitude equals the rate of change in that direction.



Divergence Operator

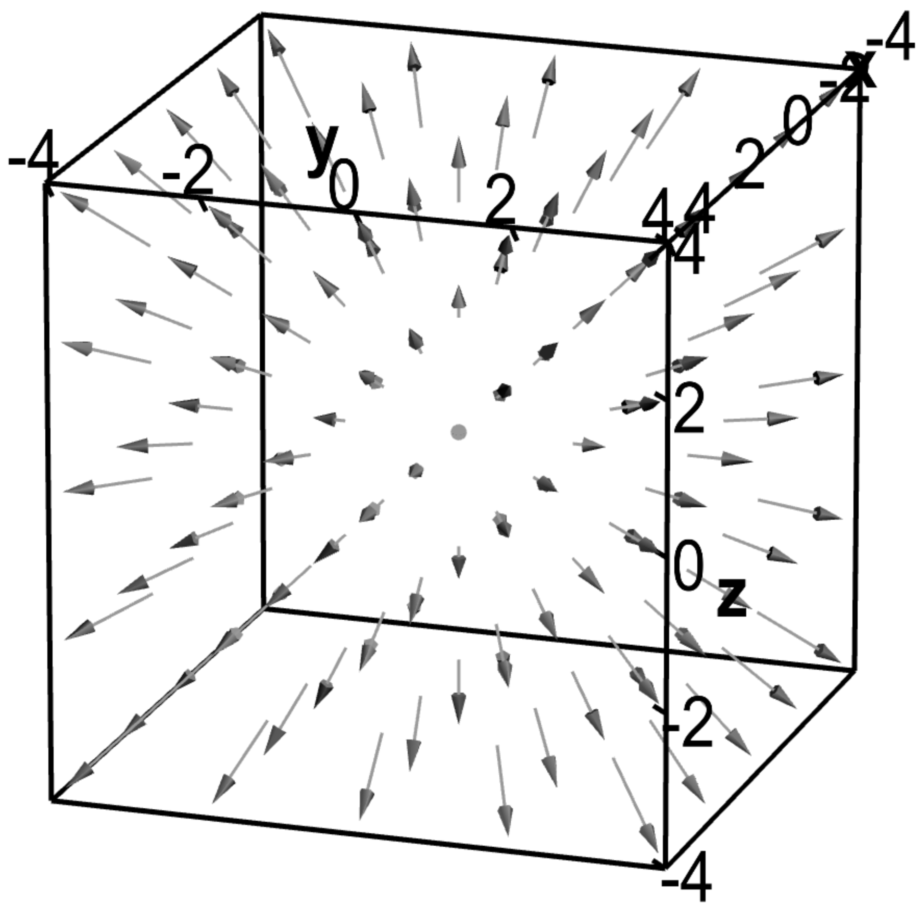
The divergence operator has the same form as the gradient, but has the opposite effect on the rank of the quantity on which it operates. Applied to a vector it produces a scalar; applied to a tensor it produces a vector.

$$\nabla \cdot \vec{b} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [b_x, b_y, b_z] = \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}$$

Divergence Operator

The divergence of a vector field may be thought of as the local rate of expansion of the vector field.

The physical significance of the divergence of a vector field is the rate at which “density" exits a given region of space.



Curl Operator

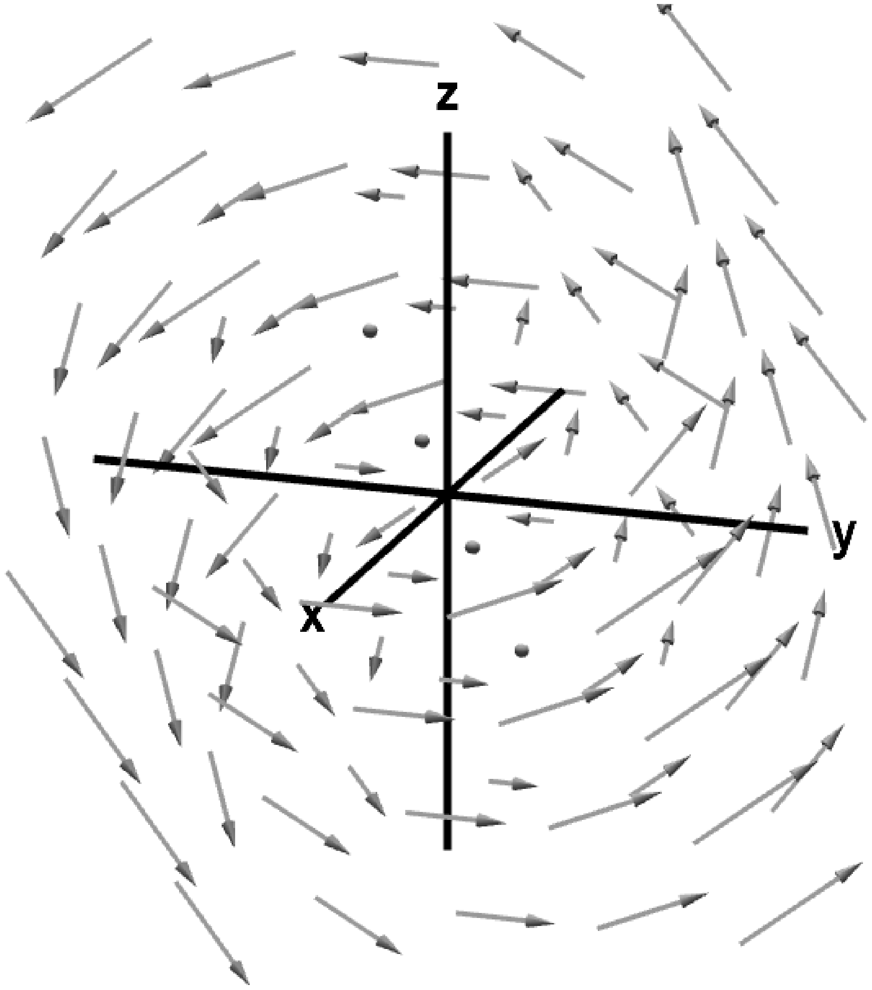
Finally, the curl is a matrix operating on a vector field:

$$\nabla \times \vec{b} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_x & b_y & b_z \end{bmatrix} = \begin{bmatrix} \frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z} \\ \frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x} \\ \frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} \end{bmatrix}$$

Curl Operator

The curl may be thought of as the local curvature of the vector field.

The physical significance of the curl of a vector field is the amount of “rotation" or angular momentum of the contents of a given region of space



Vector Identities

There are various identities in vector calculus that are useful in seismology. Here are some:

$$\nabla \times \nabla f = 0$$

$$\nabla (f \vec{v}) = f \nabla \cdot \vec{v} + \vec{v} \cdot \nabla f$$

$$\nabla \cdot \nabla \times \vec{v} = 0$$

$$\nabla (fg) = f \nabla g + g \nabla f$$

$$\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \times \vec{u} - \vec{u} \cdot \nabla \times \vec{v}$$

$$\nabla \times (f \vec{v}) = \nabla f \times \vec{v} + f \nabla \times \vec{v}$$

$$\nabla^2 \vec{u} = \nabla (\nabla \cdot \vec{u}) - \nabla \times \nabla \times \vec{u}$$