Engineering Seismology and Seismic Hazard – 2019

Lecture 6

Elastic Theory

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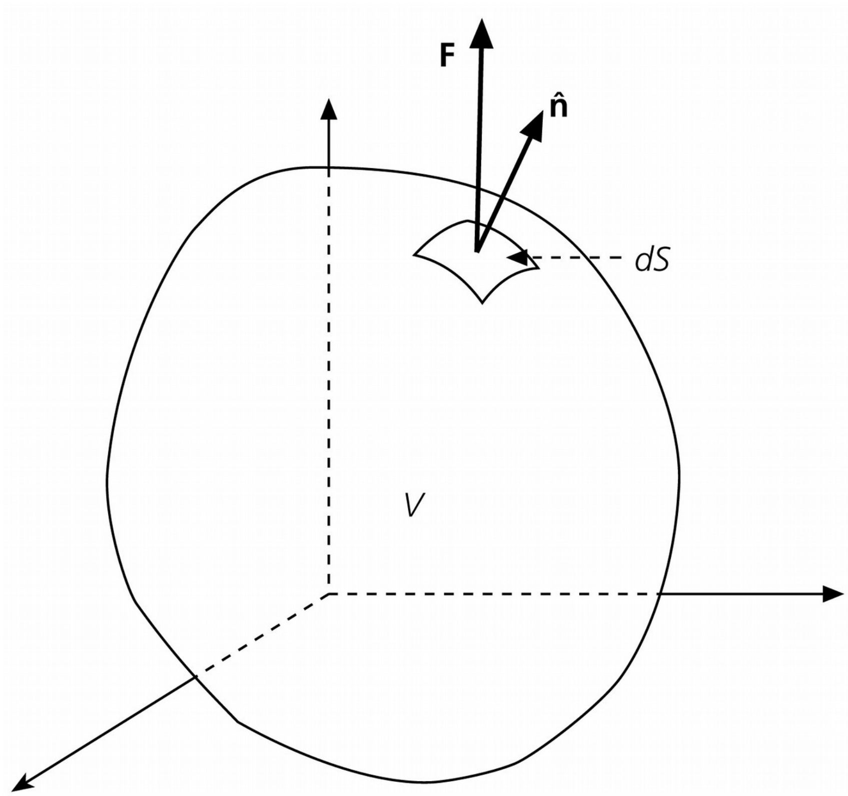
Body and Surface Forces

There are essentially two types of forces acting on a finite volume:
1) Body forces, such as gravity
2) Surface forces, such as atmospheric pressure



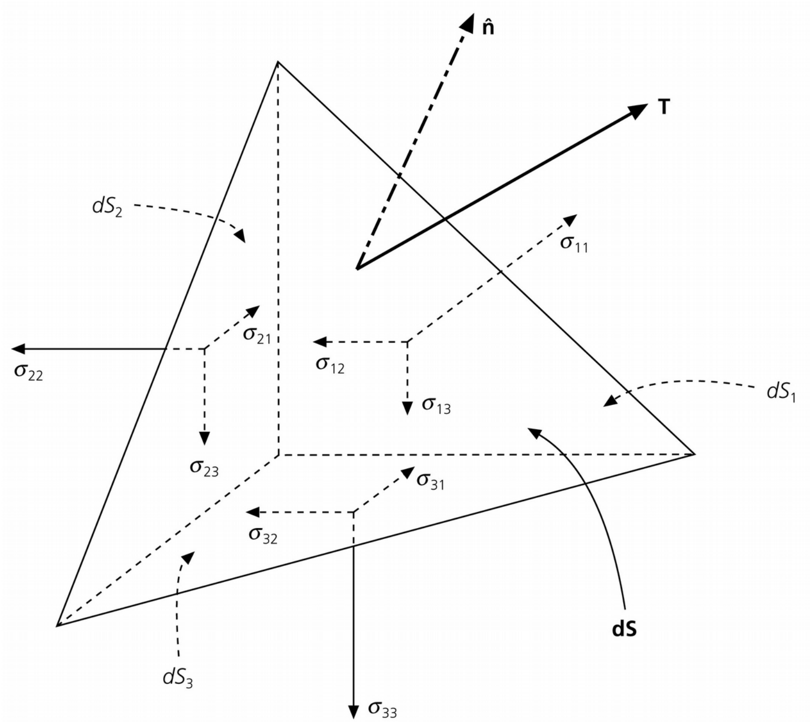
Traction vector:

$$\vec{T}(\hat{n}) = \lim_{dS \to 0} \frac{\vec{F}}{dS}$$

$$\vec{T}(\hat{n}) = T_1 \hat{x}_1 + T_2 \hat{x}_2 + T_3 \hat{x}_3$$

Static Equilibrium Condition

Let us now assume a volume with an tetrahedral shape. On each face, a traction vector is acting.



In the static case, equilibrium condition requires zero balance of all forces:

$$\int \vec{F} = 0$$

Which is, in term of tractions:

$$\vec{T}(\hat{n})\, dS_n - \vec{T}(\hat{x}_1)\, dS_1 - \vec{T}(\hat{x}_2)\, dS_2 - \vec{T}(\hat{x}_3)\, dS_3 = 0$$

Cauchy's Equation

By using the identity:

$$\frac{dS_i}{dS_n} = \hat{n} \cdot \hat{x}_i = n_i$$

It is possible to write the equilibrium equation as:

$$\vec{T}(\hat{n}) = \vec{T}(\hat{x}_1) n_1 + \vec{T}(\hat{x}_2) n_2 + \vec{T}(\hat{x}_3) n_3$$

Or, identically by components:

$$T_1(\hat{n}) = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$
$$T_2(\hat{n}) = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$
$$T_3(\hat{n}) = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

$$\Rightarrow \quad T_i(\hat{n}) = \sum_{j=1}^{3} \sigma_{ij} n_j$$

The Stress Tensor

The Chaucy's equation can be represented in a more compact matrix form.

$$\begin{bmatrix} T_1(\hat{n}) \\ T_2(\hat{n}) \\ T_3(\hat{n}) \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

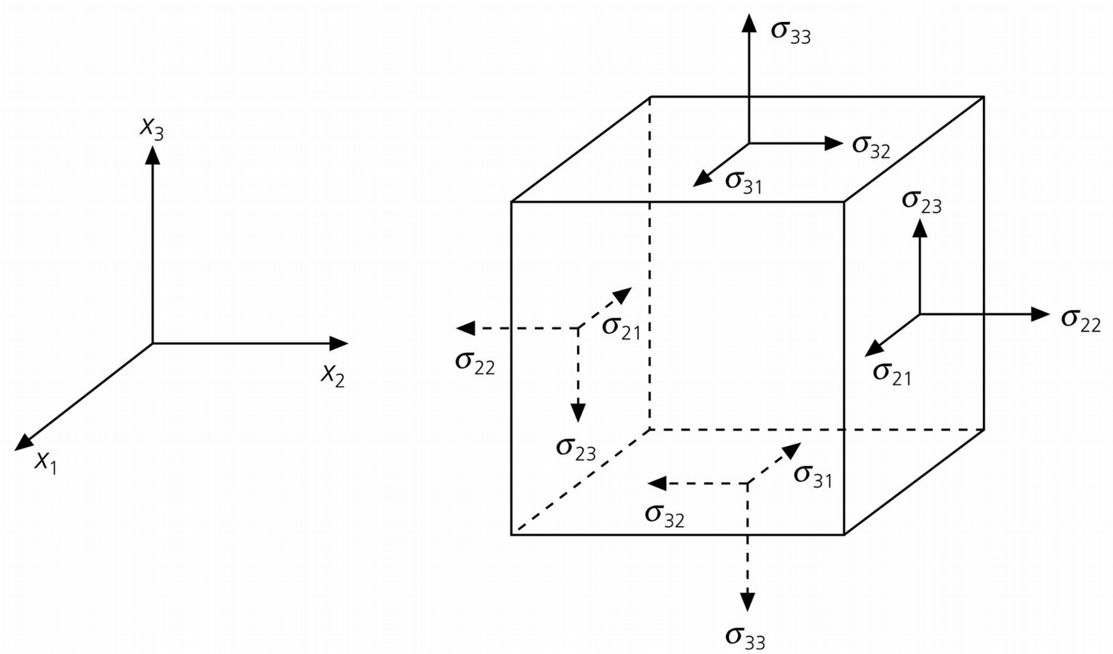
$$\boxed{\vec{T} = \boldsymbol{\sigma} \cdot \hat{n}}$$

Where σ is the Stress Tensor

Properties of the Stress Tensor

Normal stress: i = j
Shear stress: i ≠ j

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



The stress tensor is symmetric, therefore the stress state of a surface of arbitrary orientation is fully described by only 6 independent components

$$\sigma_{ij} = \sigma_{ji}$$

Principal Stresses

For a given stress state it is possible to find a rotation of the reference system for which only normal stress are present, while shear components vanish.

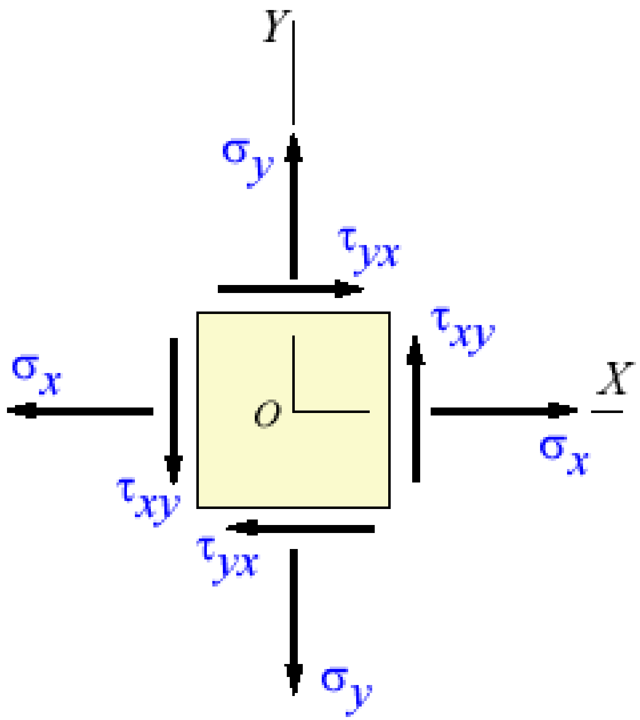
Axis of the new system are the principal directions, while the corresponding normal stresses are the principal stresses.

Principal directions and stresses can be found by diagonalisation of the stress tensor using eigenvalue decomposition:

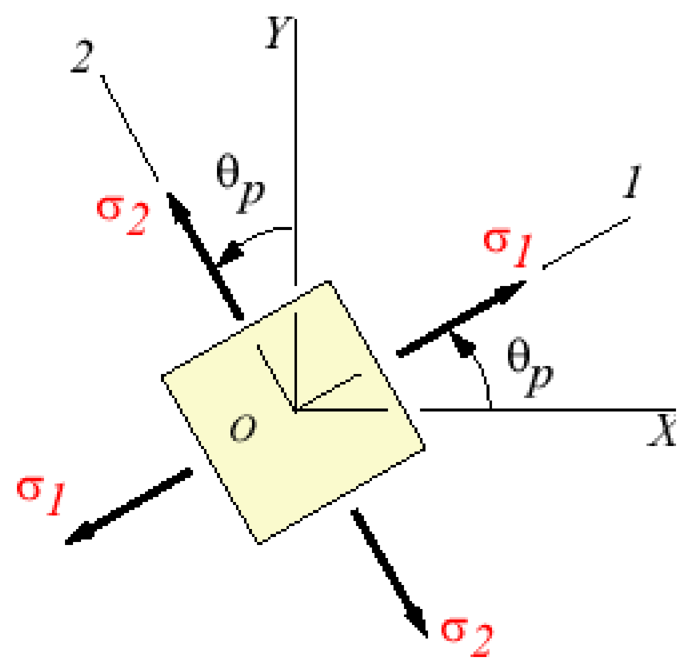
$$(\sigma_{ij} - \lambda \, \delta_{ij}) n_j = 0$$

$$\begin{bmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

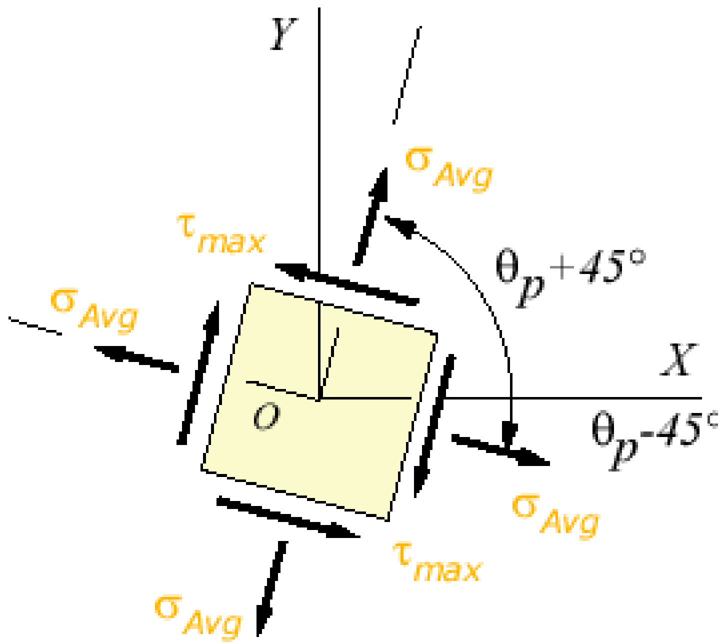
Principal Directions



Original state



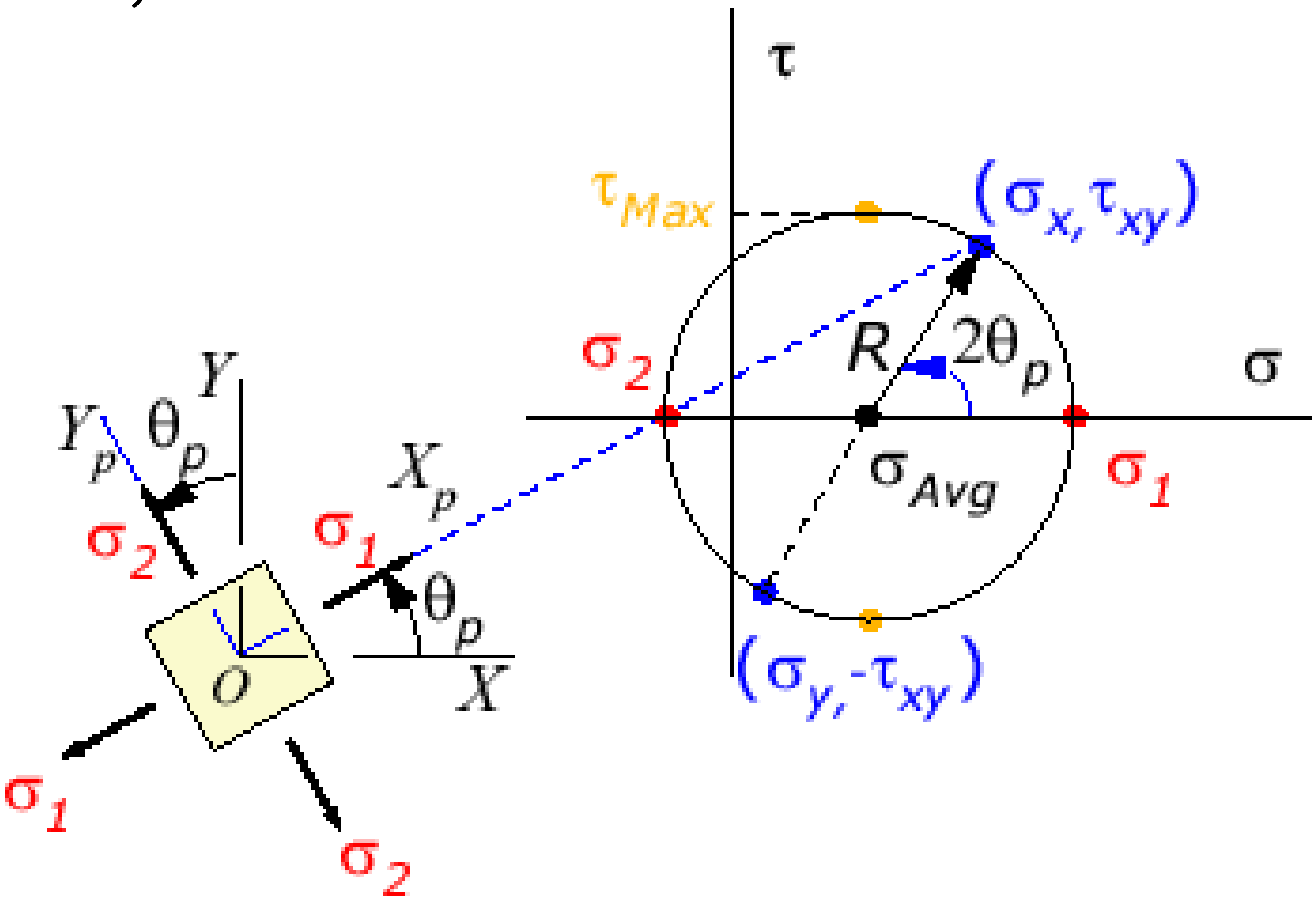
Principal directions



Maximum shear directions

Mohr's Circle

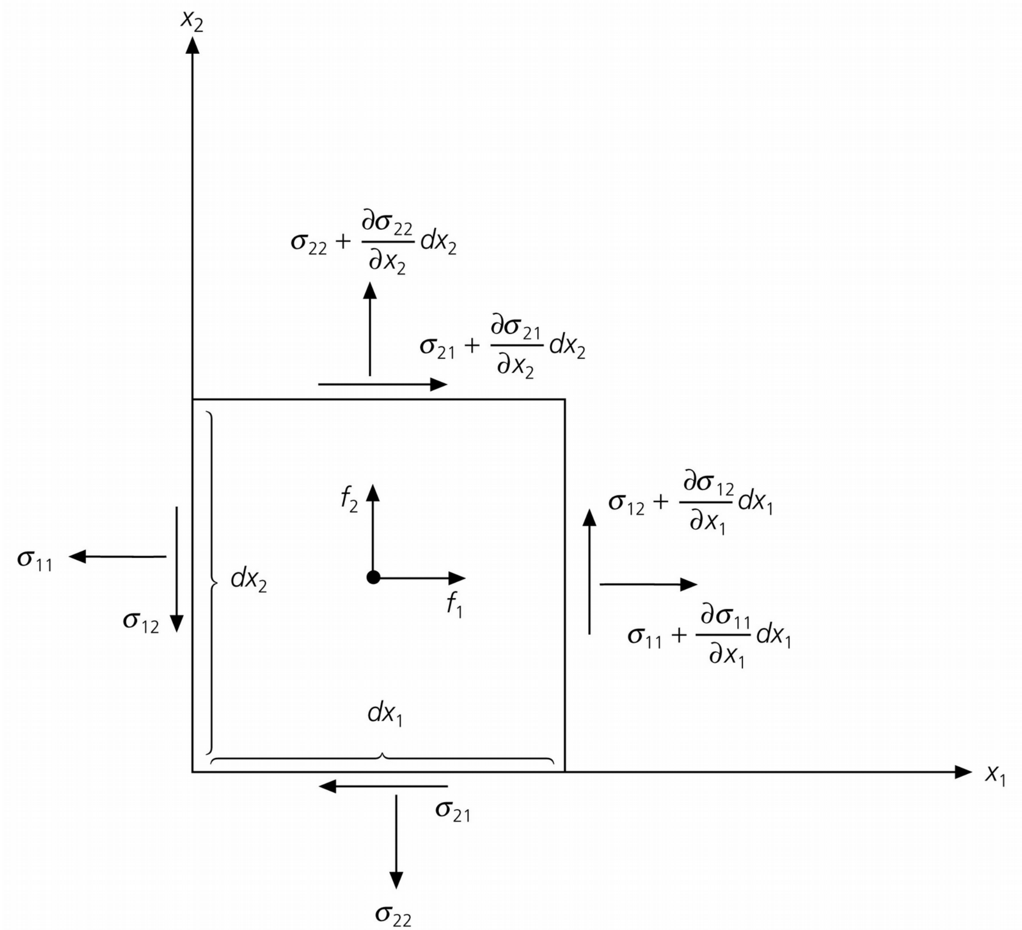
Principal directions can also be conveniently analyzed using Mohr's circle representation, which is also used to study fracturing (more on this later)



Static Equilibrium using Stress

We now repeat the previous exercise for a different volume shape.

In such case, force equilibrium can be computed by component separately as:



$$F_1^{TOT} =$$
$$\sigma_{11}^{dx_1} dx_2 dx_3 - \sigma_{11}^{0} dx_2 dx_3 +$$
$$\sigma_{21}^{dx_2} dx_1 dx_3 - \sigma_{21}^{0} dx_1 dx_3 +$$
$$\sigma_{31}^{dx_3} dx_1 dx_2 - \sigma_{31}^{0} dx_1 dx_2 = 0$$

Static Equilibrium using Stress

By truncated expansion in Taylor series, stresses can be written as:

$$\sigma_{11}^{dx_1} = \sigma_{11}^{0} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1$$

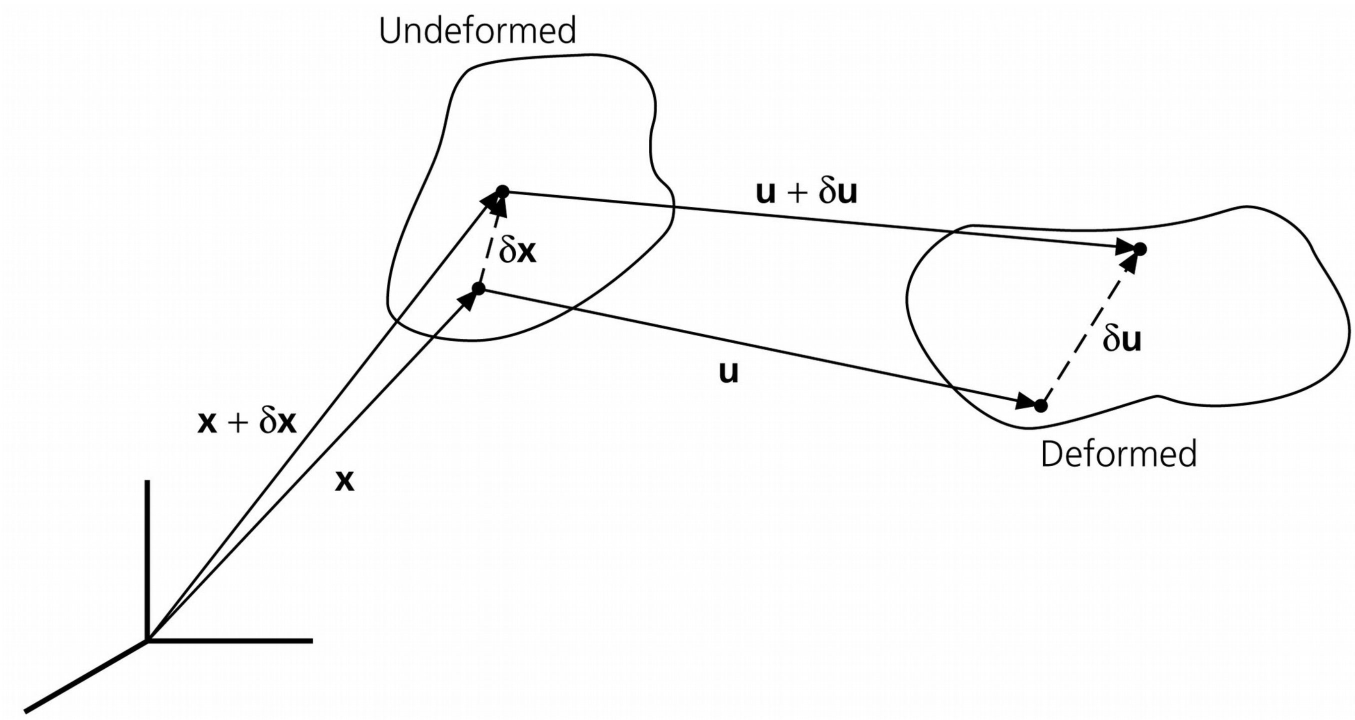
By substitution, the equilibrium equation can be written in the form:

$$F_1 = \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} \right) dx_1 dx_2 dx_3 = 0$$

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} = 0 \quad \Rightarrow \quad \sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} + f^V = 0$$

Displacement Vector

In a body subject to displacement (u), deformation can be obtained by the relative variation of displacement between adjacent points



$$du_j(x_i) = u_j(x_i + dx_i) - u_j(x_i) = \frac{\partial u_j}{\partial x_i} dx_i$$

Strain and Rotation Tensors

Rearranging elements...

$$du_j(x_i) = \frac{\partial u_j}{\partial x_i} dx_i + \frac{1}{2}\frac{\partial u_j}{\partial x_i} dx_i - \frac{1}{2}\frac{\partial u_i}{\partial x_j} dx_i$$

$$du_j(x_i) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) dx_i + \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) dx_i$$

$$\frac{du_j(x_i)}{dx_i} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$$

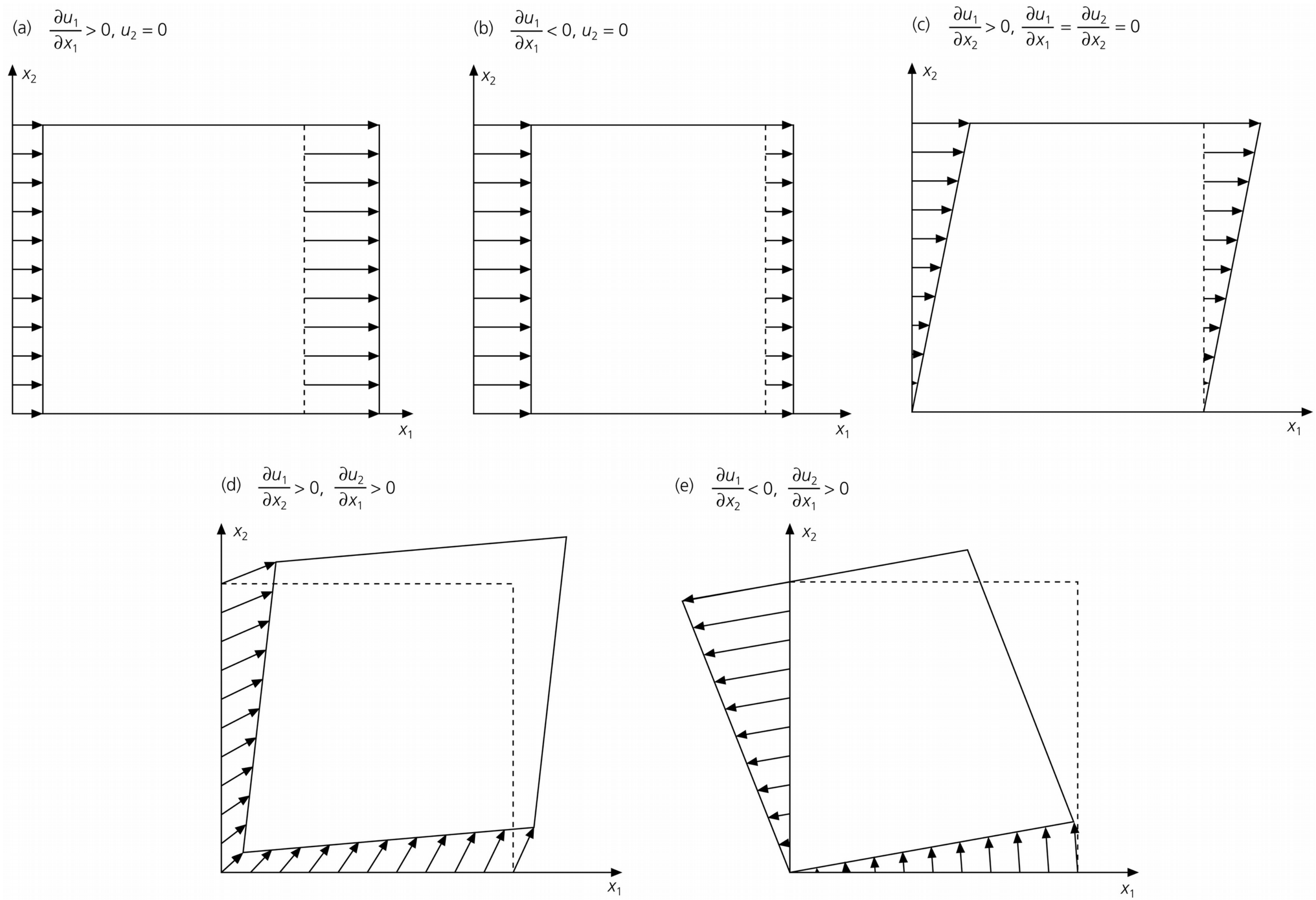
Strain and Rotation Tensors

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \rightarrow \text{Strain Tensor}$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) \rightarrow \text{Rotation Tensor}$$

While the strain tensor is symmetrical, rotation tensor is anti-symmetrical

Examples of Deformation



Hooke's Law

The relation between stress and deformation is expressed by the Hook's law:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

Where C is the fourth-order **stiffness tensor**.

In case of isotropic material, due to symmetries and energetic considerations, the tensor reduces from 81 elements to just two constants, the **Lame's parameters** λ and μ:

$$C_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right)$$

Hooke's Law

The Hook's law for the isotropic case can then be simplified as:

$$\sigma_{ij} = \lambda\ \delta_{ji}\,\varepsilon_{kk} + 2\mu\,\varepsilon_{ij}$$

Or, identically as function of displacement:

$$\sigma_{ij} = \lambda\ \delta_{ji}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

Elastodynamic Equilibrium

Now we examine the case when the material is subject to some acceleration and find in dynamic condition:

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

By using the Hooke's law and after some maths, the elastodynamic equation above can be written in displacement as:

$$(\lambda + \mu) \frac{\partial^2 u_k}{\partial x_i \partial x_k} \delta_{ji} + \mu \frac{\partial^2 u_i}{\partial x_j^2} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

Equation of Motion in Vector Form

The wave equation can be written in a more compact and general for in vector notation:

$$(\lambda + 2\mu)\nabla(\nabla \cdot \vec{u}) - \mu\nabla\times\nabla\times\vec{u} = \rho\frac{\partial^2\vec{u}}{\partial t^2}$$

VOLUMETRIC DEFORMATION

ANGULAR DEFORMATION

This notation is also called the **Navier's equation**

(Generic) Wave Equation Solution

Let us consider a generic hyperbolic partial differential equation (scalar wave equation) with velocity *c*:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

To solve it, we can assume a generic solution (in Fourier domain):

$$\mathrm{u}(x,t) = G(x)e^{-i\omega t} \quad \Rightarrow \quad \frac{\partial^2 G(x)}{\partial x^2} = -\frac{\omega^2}{c^2}G(x)$$

Which is the eigenvalue function for *G(x), with solution:*

$$\mathrm{G}(x) = Ae^{\mp i\kappa x} \quad \Rightarrow \quad \boxed{\mathrm{u}(x,t) = A\,e^{-i(\omega t \pm \kappa x)}}$$

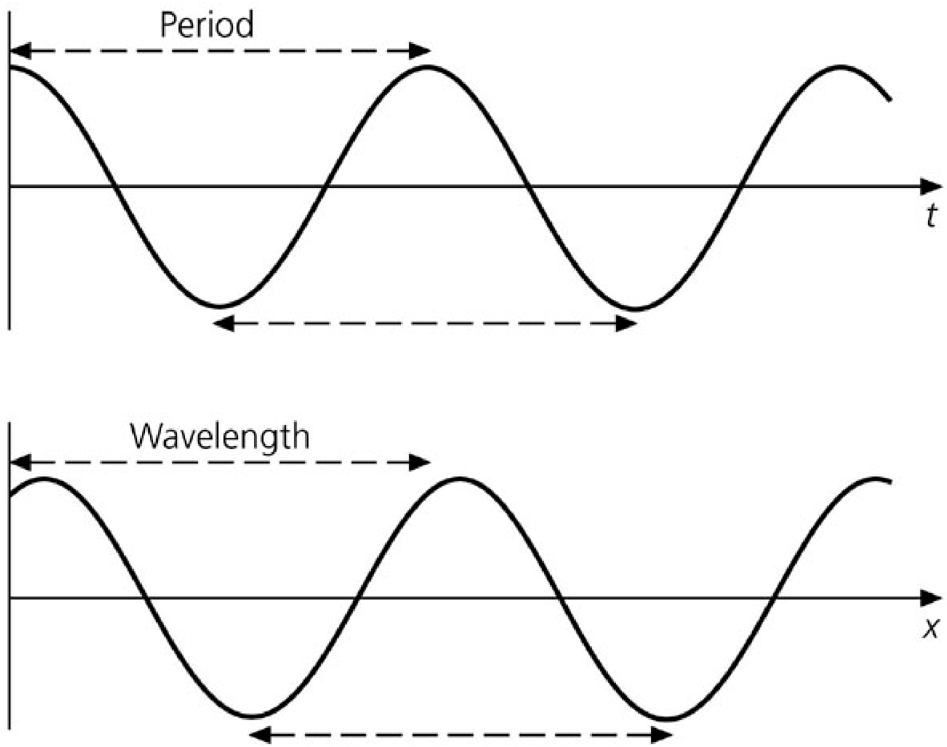
Plane Waves

The solution is actually the real part of the harmonic equation. Using the Euler formula, we can rewrite the solution as:

$$\mathrm{u}(x,t)=A[\cos(\omega t\pm\kappa x)-i\sin(\omega t\pm\kappa x)]$$

With:

$$c=\frac{\omega}{\kappa}=\frac{\lambda}{T}$$



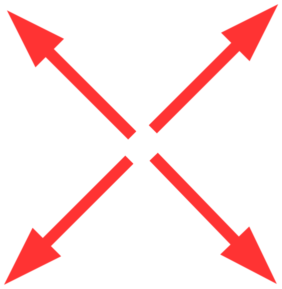
Helmholtz Theorem

Any vector field u=u(x) may be separated into a scalar and a vector potential:

$$\vec{u} = \nabla \phi + \nabla \times \vec{\psi}$$

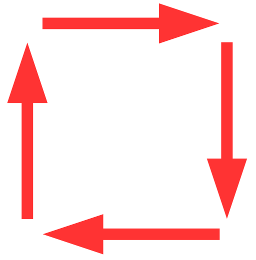
The scalar field is irrotational (does not rotate, no angular distortion):

$$\nabla \times \phi = 0$$



While the vector field is solenoidal (does not diverge, same volume):

$$\nabla \cdot \vec{\psi} = 0$$



Wave Equation using Potentials

By substitution of Helmholtz potentials into the Navier's equation we get (note that null terms are neglected):

$$(\lambda+2\mu)\nabla(\nabla\cdot(\nabla\phi))-\mu\nabla\times\nabla\times(\nabla\times\vec{\psi})=\frac{\partial^2(\nabla\phi+\nabla\times\vec{\psi})}{\partial t^2}$$

And after some rearranging:

$$\nabla\left((\lambda+2\mu)\nabla^2\phi-\rho\frac{\partial^2\phi}{\partial t^2}\right)+\nabla\times\left(\mu\nabla^2\vec{\psi}-\rho\frac{\partial^2\vec{\psi}}{\partial t^2}\right)=0$$

Wave Equation using Potentials

Such equation is satisfied for:

$$(\lambda + 2\mu)\nabla^2 \phi = \rho \frac{\partial^2 \phi}{\partial t^2}$$

$$\mu \nabla^2 \vec{\psi} = \rho \frac{\partial^2 \vec{\psi}}{\partial t^2}$$

Which are standard Hyperbolic partial differential equations with velocities:

$$\alpha = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}$$

$$\beta = \sqrt{\frac{\mu}{\rho}}$$

$$\alpha > \beta$$

Wave Solution using Potentials

The harmonic solution for the scalar and vector potentials is:

$$\phi = A e^{i(\omega t - \vec{\kappa}_\alpha \cdot \vec{x})} \qquad \vec{\psi} = \hat{n} B e^{i(\omega t - \vec{\kappa}_\beta \cdot \vec{x})}$$

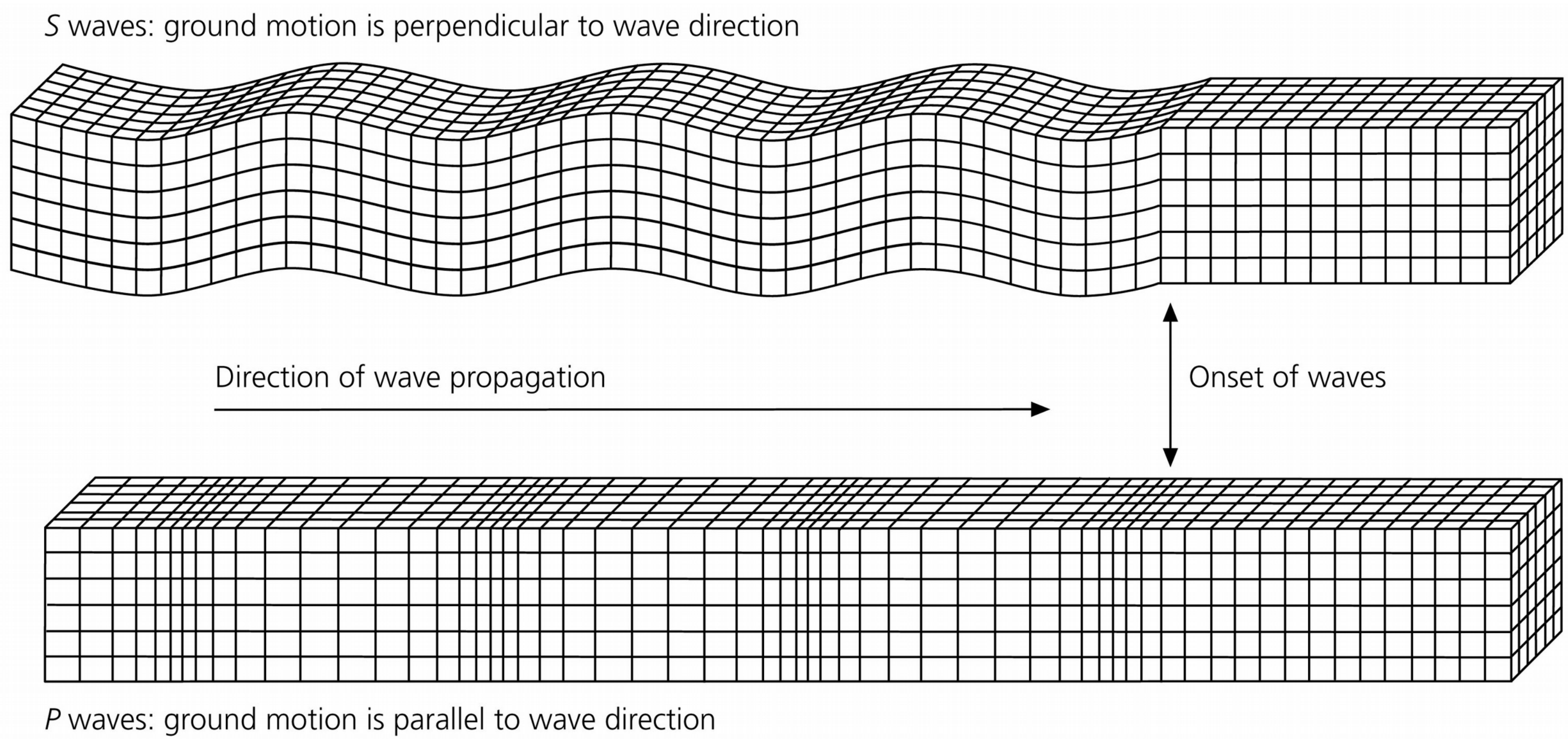
By substitution into the Helmholtz equation, we get the generic harmonic solution of Nevier's equation:

$$\vec{u} = \underbrace{\nabla \left(A e^{i(\omega t - \vec{\kappa}_\alpha \cdot \vec{x})} \right)}_{\vec{u}_P} + \underbrace{\nabla \times \left(\hat{n} B e^{i(\omega t - \vec{\kappa}_\beta \cdot \vec{x})} \right)}_{\vec{u}_S}$$

Or identically:

$$\vec{u} = \left(A \nabla e^{-i \vec{\kappa}_\alpha \cdot \vec{x}} + B \nabla \times \hat{n} e^{-i \vec{\kappa}_\beta \cdot \vec{x}} \right) e^{i \omega t}$$

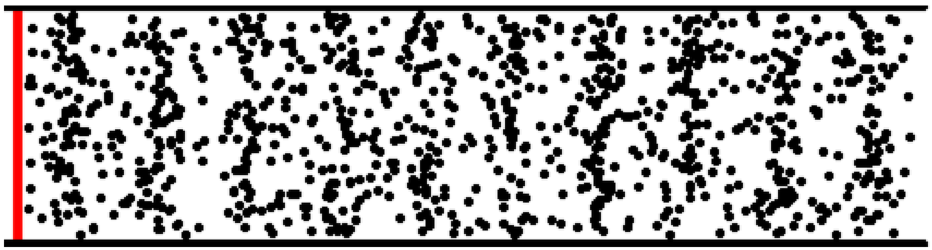
P and S waves



The wave equation solution leads then to two types of waves:
1) An irrotational wave (no angular distortions), called P (*primae*)
2) an equivoluminal wave (no change in volume), called S (*secundae*)

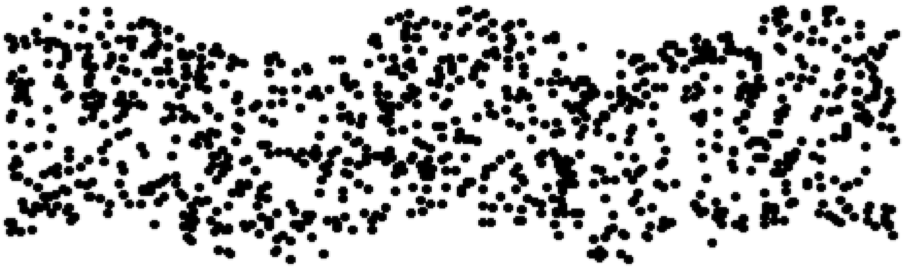
P and S waves

P



(from Dr. Dan Russel, University of Kettering – USA)

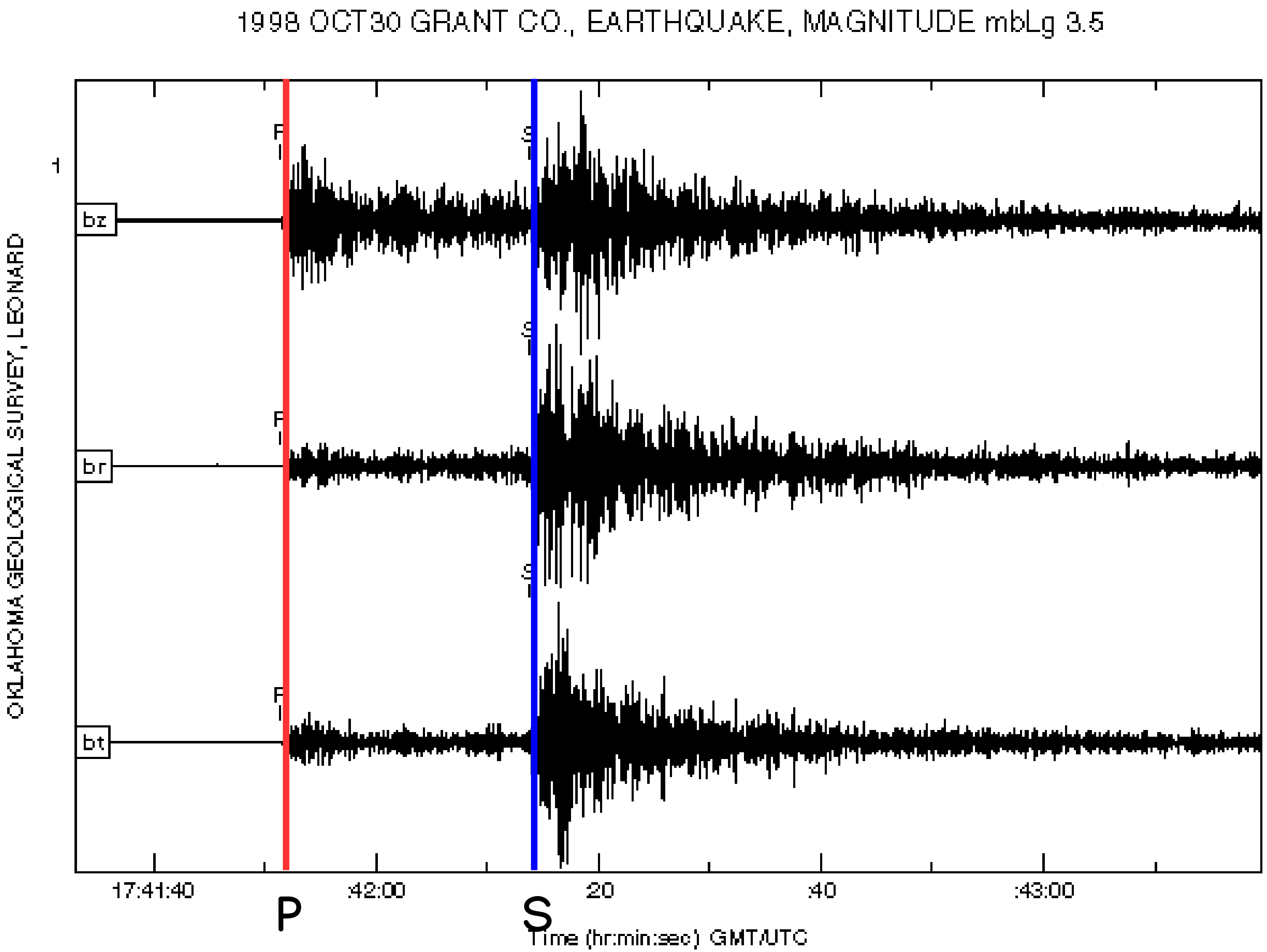
S



Longitudinal P waves propagate both in fluids and in solids. Transversal S waves ONLY in solids (fluids has no shear resistance).

P and S waves are non-dispersive as V_P and V_S are frequency-independent.

P-S wave Arrivals



P–S wave Arrivals