Engineering Seismology and Seismic Hazard – 2019

Lecture 11

Inverse Problems

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Inversion of a linear system

Let's assumed the linear problem:

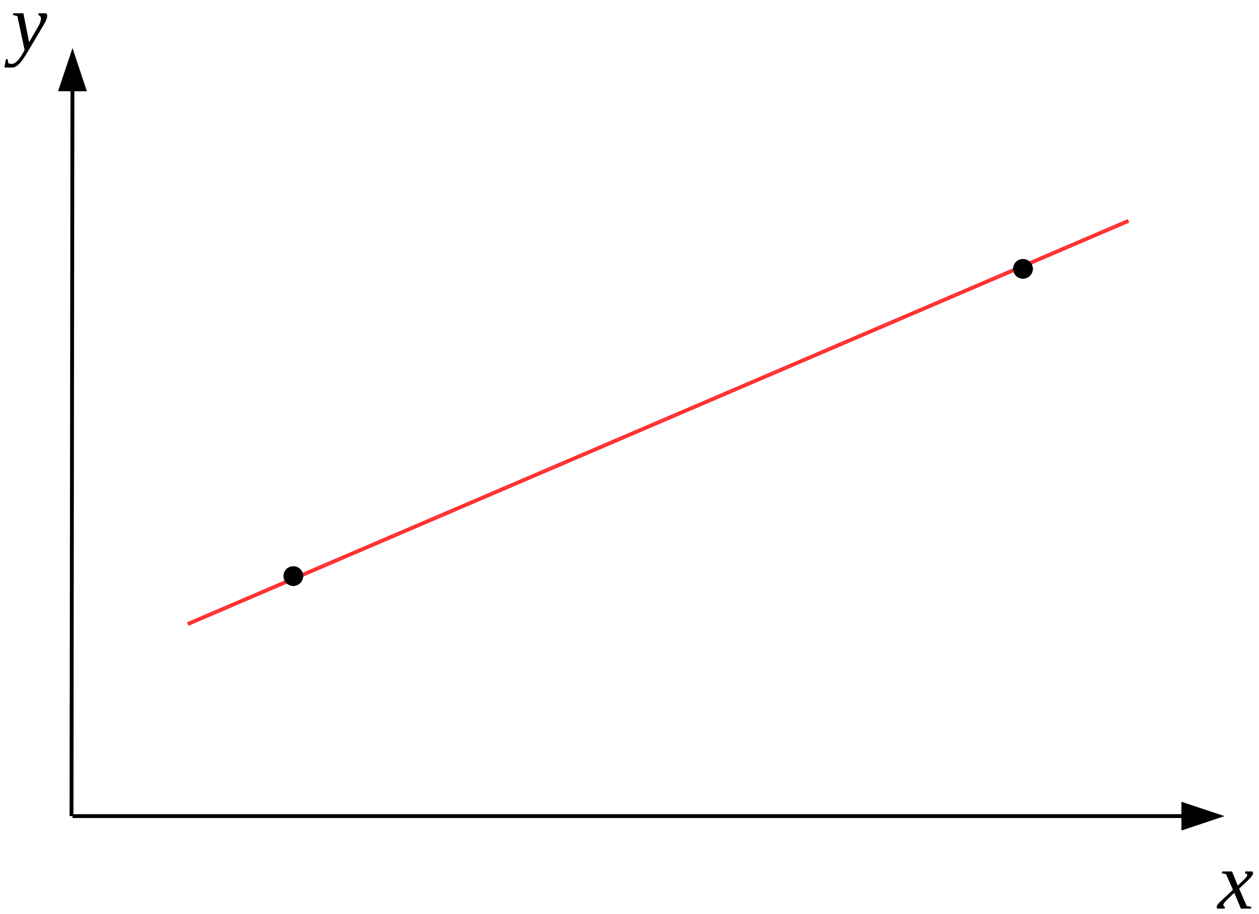
$$y = ax + b$$

The coefficients a and b can be found if at least two observations (x,y) are available. In such case, the linear system can easily be solved:

$$\begin{cases} y_1 = ax_1 + b \\ y_2 = ax_2 + b \end{cases} \quad \Rightarrow \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\vec{d} = \mathbf{G}\vec{m} \quad \Rightarrow \quad \boxed{\vec{m} = \mathbf{G}^{-1}\vec{d}}$$

Graphical Representation



Adding Data Uncertainty

Unfortunately, if data are affected by **uncertainty**, such solution provides a wrong estimated of the model parameters.

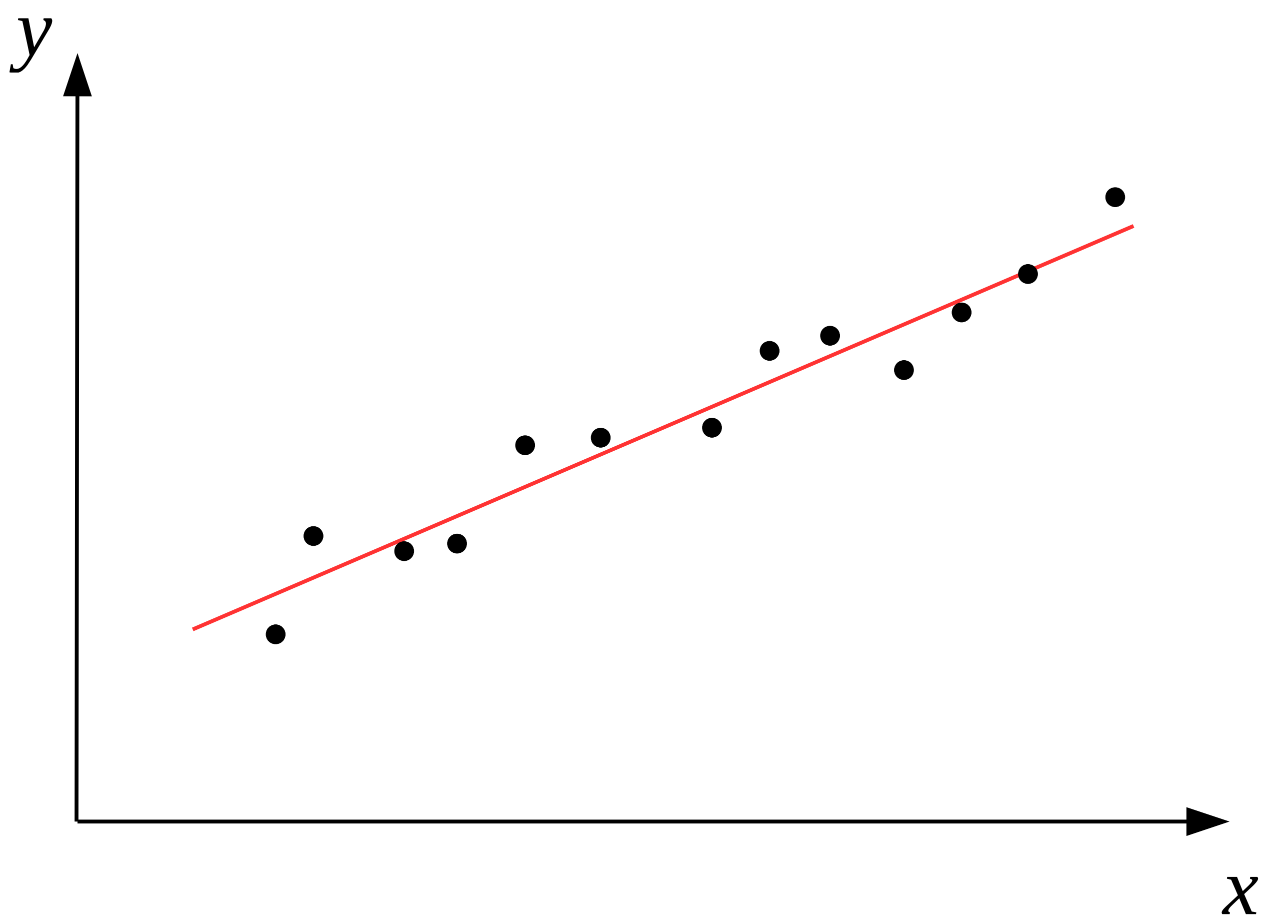
$$y = ax + b + e$$

To minimize the error, redundancy of observations is required.

$$\begin{cases} y_1 = ax_1 + b + e_1 \\ y_2 = ax_2 + b + e_2 \\ y_3 = ax_3 + b + e_3 \\ y_4 = ax_4 + b + e_4 \\ \dots \end{cases} \quad \Rightarrow \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \dots \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \dots \end{bmatrix}$$

In this case, however, the matrix G is not square anymore, and cannot be simply inverted to get a solution

Graphical Representation



Minimizing the Error

In this case, the solution must be found by minimizing the error between observations and the theoretical model to fit, e.g. minimizing its euclidean distance (sum of squares of residuals):

$$E = \|\vec{d} - \mathbf{G}\,\vec{m}\|_{min}$$

where is:

$$\|\vec{d} - \mathbf{G}\,\vec{m}\| = (\vec{d} - \mathbf{G}\,\vec{m})^T (\vec{d} - \mathbf{G}\,\vec{m})$$

or, rearranging elements:

$$\|\vec{d} - \mathbf{G}\,\vec{m}\| = \vec{d}^T \vec{d} - \vec{d}^T \mathbf{G}\,\vec{m} - \vec{m}^T G^T \vec{d} + \vec{m}^T G^T G\,\vec{m}$$

Least Square Solution

To solve that we pose the partial derivatives with respect to model parameters to vanish:

$$\frac{\delta E}{\delta \vec{m}} = \frac{\delta \|\vec{d} - \mathbf{G}\vec{m}\|}{\delta \vec{m}} = 0$$

Using the differentiation rules, this leads to:

$$0 - (\vec{d}^T G)^T - G^T \vec{d} + (G^T G \vec{m} + \vec{m}^T (G^T G)^T) = 0$$

$$-2 G^T \vec{d} + 2 G^T G \vec{m} = 0$$

$$\vec{m} = (G^T G)^{-1} G^T \vec{d}$$

Non Linear Problems

When the problem to solve is non linear, model coefficients and parameters cannot be separated. To problem has the following general form:

$$\vec{d} = g(\vec{m})$$

Note that function g is now function of model parameters m.
An example of such a problem is:

$$\begin{cases} y_1 = ae^{bx_1} + e_1 \\ y_2 = ae^{bx_2} + e_2 \\ y_3 = ae^{bx_3} + e_3 \\ y_4 = ae^{bx_4} + e_4 \\ \dots \end{cases} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \dots \end{bmatrix} = \begin{bmatrix} ae^{bx_1} \\ ae^{bx_2} \\ ae^{bx_3} \\ ae^{bx_4} \\ \dots \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \dots \end{bmatrix}$$

Jacobean Matrix

If the inverse problem is just weakly non linear, a small perturbation of an initial guess model (0) will lead to:

$$\delta \vec{d} = \vec{d} - \vec{d}_0 = g(\vec{m}) - g(\vec{m}_0) \approx \frac{\partial g(\vec{m})}{\partial \vec{m}} \delta \vec{m}$$

The matrix of partial derivatives of the model is called Jacobean, and can be expressed as:

$$\boldsymbol{J}_{\boldsymbol{ij}} = \frac{\partial g_i}{\partial m_j}$$

By substitution, we will get a more familiar expression using matrix multiplication:

$$\delta \vec{d} = \boldsymbol{J} \, \delta \vec{m}$$

Gradient Method

Given an initial guess model (0) with corresponding residual to observations, we would like to find now a new model (N) with smaller residual. To do that we minimize the following:

$$E = \left\| \left(\vec{d}_{obs} - \vec{d}_0 \right) - \left(g(\vec{m}_N) - g(\vec{m}_0) \right) \right\|_{min}$$

$$E = \left\| \delta \vec{d}_{(obs-0)} - J \, \delta \vec{m}_{(N-0)} \right\|_{min}$$

We then can apply the same considerations for the linear case:

$$\delta \vec{m}_{(N-0)} = \left(J^T J \right)^{-1} J^T \delta \vec{d}_{(obs-0)}$$

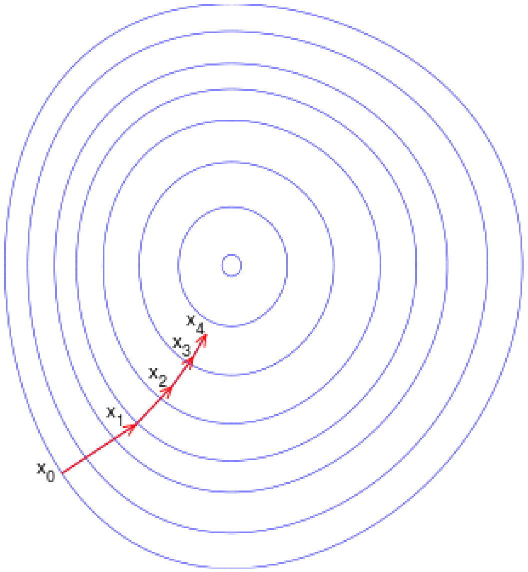
Iterative Solution

Practically, the new model can be obtained as:

$$\vec{m}_N \approx \vec{m}_0 + \left(J^T J\right)^{-1} J^T \left(\vec{d}_{obs} - \vec{d}_0\right)$$

We can iterate the process several times by assign the new model parameters as the guess model at the end of each iteration, till the difference between new and guess models is less than a defined (arbitrary) threshold:

$$\vec{m}_0 = \vec{m}_N \Rightarrow \delta \vec{m} \leq E$$



Heavily Non Linear Problem

For heavy non linear problems, linearisation might not be possible. A brute-force approach is often used instead.

In this case the problem is solved by directly comparing all possible models (or a realistic subset of it) obtained from the combination of the different model parameters.

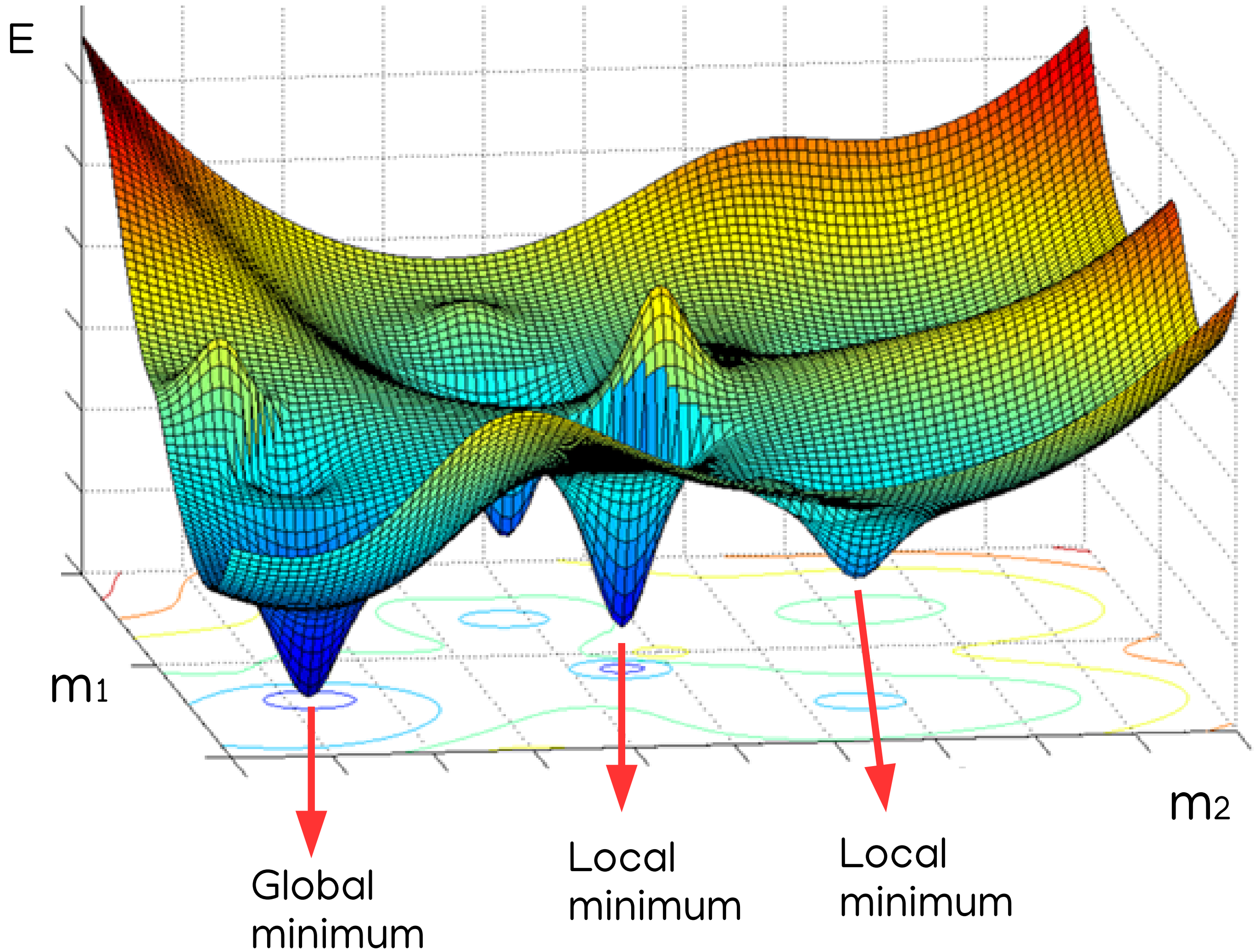
The best-fitting model is then selected as the one with lowest residual with respect to observation:

$$E = \| \vec{d}_{obs} - \vec{d}_{mod} \|_{min}$$

This is also called the misfit function.

Local vs Global Minima

The inversion problem might be non-unique, and the solution might be trapped in a local minima of the misfit function.



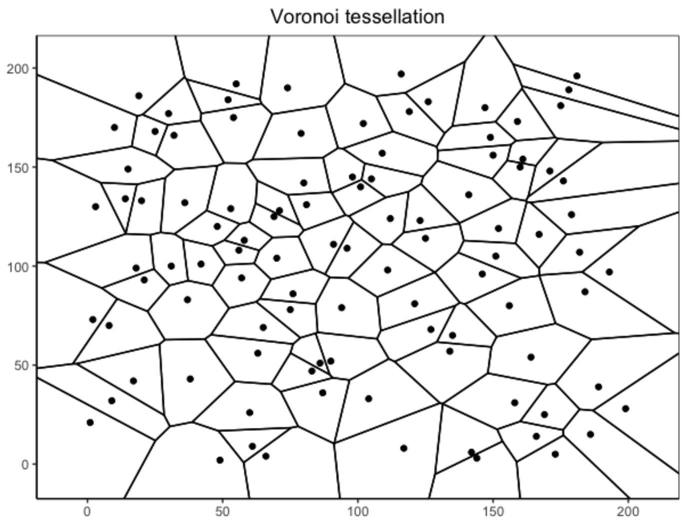
Parameter Sampling

Testing all possible models might not be possible in reality, particularly because:

1) model parameters are continuous variable
2) the model depends on many parameters

To overcome this, smart sampling strategy of the parameter space are used, such as:

- Montecarlo sampling
- Simulated annealing
- Genetic algorithms
- Neighborhood algorithms



Moreover a-priori information can be used, such as limiting the range of variability of the model parameters to realistic values (parameter bounding)